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FOREIGN TECHNOLOGY DIVISION

CONTACT PROBLEM OF THE THEORY OF ELASTICITY

by

I. Ya. Shtayerman

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CONTACT PROBLEM OF THE THEORY OF ELASTICITY

By: I. Ya. Shtayerman

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U. S. BOARD ON GEOGRAPHIC NAMES TRANSLITERATION SYSTEM

Block	Italic	Transliteration	Block	Italic	Transliteration
А а	*А а*	A, a	Р р	*Р р*	R, r
Б б	*Б б*	B, b	С с	*С с*	S, s
В в	*В в*	V, v	Т т	*Т т*	T, t
Г г	*Г г*	G, g	У у	*У у*	U, u
Д д	*Д д*	D, d	Ф ф	*Ф ф*	F, f
Е е	*Е е*	Ye, ye; E, e*	Х х	*Х х*	Kh, kh
Ж ж	*Ж ж*	Zh, zh	Ц ц	*Ц ц*	Ts, ts
З з	*З з*	Z, z	Ч ч	*Ч ч*	Ch, ch
И и	*И и*	I, i	Ш ш	*Ш ш*	Sh, sh
Й й	*Й й*	Y, y	Щ щ	*Щ щ*	Shch, shch
К к	*К к*	K, k	Ъ ъ	*Ъ ъ*	"
Л л	*Л л*	L, l	Ы ы	*Ы ы*	Y, y
М м	*М м*	M, m	Ь ь	*Ь ь*	'
Н н	*Н н*	N, n	Э э	*Э э*	E, e
О о	*О о*	O, o	Ю ю	*Ю ю*	Yu, yu
П п	*П п*	P, p	Я я	*Я я*	Ya, ya

* ye initially, after vowels, and after ъ, ь; e elsewhere. When written as ё in Russian, transliterate as yё or ё. The use of diacritical marks is preferred, but such marks may be omitted when expediency dictates.

FOLLOWING ARE THE CORRESPONDING RUSSIAN AND ENGLISH DESIGNATIONS OF THE TRIGONOMETRIC FUNCTIONS

Russian	English
sin	sin
cos	cos
tg	tan
ctg	cot
sec	sec
cosec	csc
sh	sinh
ch	cosh
th	tanh
cth	coth
sch	sech
csch	csch
arc sin	$\sin^{-1}$
arc cos	$\cos^{-1}$
arc tg	$\tan^{-1}$
arc ctg	$\cot^{-1}$
arc sec	$\sec^{-1}$
arc cosec	$\csc^{-1}$
arc sh	$\sinh^{-1}$
arc ch	$\cosh^{-1}$
arc th	$\tanh^{-1}$
arc cth	$\coth^{-1}$
arc sch	sech^{-1}
arc csch	csch^{-1}

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rot	curl
lg	log

PREFACE

The contact problem is one of the basic problems in the theory of elasticity. Calculation of the many important parts, structures and machines is based on the theory of the compression of elastic bodies. However, this theory presents considerable mathematical difficulties. The first correct solution of the basic case of the contact problem was given by Hertz[1] as long ago as in 1882, and the mathematical development of the problem set on this solution for approximately 60 years. During this period efforts of engineers and theoreticians were directed mainly at the experimental checking of the theory and the development of its applications in engineering (works of academicians, A. N. Dinnik, N. M. Belyayev and others).

In the 1920's and especially 1930's and 1940's, the mathematical base for the solution of the contact problem became quite different from what it was in the second half of the last century. Hertz used in his investigation only formulas from the theory of potential of a uniform ellipsoid, which represents the simplest prototype of solutions of problems of the theory of potential and theory of integral equations; whereas, starting approximately from the 1930's we had available the powerful, developed by us in the Soviet Union, apparatus of the resolution of problems of the theory of elasticity,

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[1]Hertz H., Gesammelte Werke, t. 1 Leipzig, 1895, str. 155.

and the other one — in his monograph, "Singular integral equations" and in articles preceding it. In these books extensive bibliographic data can be found. Let us note that in science, to a certain degree of the related theory of elasticity, namely, in hydro- and aerodynamics, for a long time in resolution of problems about two-dimensional motion of liquid and about the lift of a wing functions of the complex variable and singular integral equations were used.

It is quite natural that in the Soviet Union a number of works have appeared in which the contact problem of the theory of elasticity has received substantial improvement and development.

For the first time solutions of new contact problems, which are a generalization of the basic case, were given by me[1]. In subsequent articles[2] I obtained the solution of a number of other problems, using partly the mathematical apparatus created by Academician A. M. Lyapunov[3].

Very valuable solutions were obtained by a number of authors, especially in the school of N. I. Muskhelishvili[4], and also by L. A. Galin[5], A. I. Lur'ye[6], G. N. Savin[7] and others. Thus, at present the theory of the contact problem has attained such great development that it can be examined as a large independent branch of the theory of elasticity, which has an important practical value for a calculation of parts of structures and machines.

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[1]Shtayerman I. Ya., K teorii Gertsa mestnykh deformatsiy pri szhatii uprugikh tel (On the theory of Hertz of local deformations with the compression of elastic bodies). Doklady AN SSSR, t. XXV, No. 5, 1939.

[2]Shtayerman I. Ya., Obobshcheniye teorii Gertsa mestnykh deformatsiy pri szhatii uprugikh tel (Generalization of the theory of Hertz of local deformations with the compression of elastic bodies) (Doklady AN SSSR, t. XXIX, No. 3, 1940). Mestnyye deformatsii pri szhatii uprugikh krugovykh tsilindrov, radiusy kotorykh pochti ravny (Local deformations with the compression of elastic circular cylinders, the radii of which are almost equal) (Doklady AN SSSR, t. XXIX, No. 3, 1940). K voprosu o mestnykh deformatsiyakh pri szhatii uprugikh tel (On the question of local deformations with the

One of the necessary prerequisites for this is the bringing of the mathematical theory of the contact problem to the theoretical engineers.

The account of our book has been given in such a manner so that it with a few exceptions is accessible to the engineer familiar with a course of higher mathematics of a technical college and having certain experience in the reading of mathematical literature.

The ~~entire~~ first chapter of ~~our~~ book is devoted to methods of the solution of fundamental equations of the contact problem. We try to combine the simplicity of the account with proper fullness of the mathematical scope.

The second part contains, together with the classical investigations, an account of certain works of Soviet mathematicians on the two-dimentional contact problem of the theory of elasticity, including my works, part of which has been published for the first time[7]. these include: a new formulation of the problem on the pressure of a stamp

[FOOTNOTE CONT'D FROM PRECEDING PAGE].

compression of elastic bodies) (Doklady AN SSSR, t. XXXI, No. 8, 1941). Nekotoryye osobyye sluchai kontaktnoy zadachi (Certain special cases of contact problem) (Doklady AN SSSR, t. XXXVIII, No. 7, 1943). Ob odnom obobshchenii zadachi Gertsa (On one generalization of the problem of Hertz) (Zhurnal "Prikladnaya matematika i mekhanika", t. 7, vyp. 3, 1941).

[3]Liapounoff A., Sur les figures d'equilibre, III chast', St. Petersbourg, 1912.

[4]See Muskhelishvili N. I., Singulyarnyye integral'nyye uravneniya (Singular integral equations), Gostekhizdat, 1946

[5]Galin L. A., Issledovaniyye smeshannykh zadach teorii uprugostm (Investigation of mixed problems of the theory of elasticity) (Doktorskaya dissertatsiya, (Doctoral Dissertation) Institut mekhaniki AN), Moskva, 1946.

[6]Lur'ye A. I., Nekotoryye kontaktnyye zadachi teorii uprugosti. Zhurnal (Certain contact problems of the theory of elasticity. Journal) "Prikladnaya mekhanika i matematika", t. V, vyp. 3, 1941.

[7]DAN URSP No. 6, 1939; No. 7, 1940; Soobshcheniya Gruzinskogo Filiala AN SSSR, t. I. No. 10, 1940 g.

on an elastic half-plane, ~~discussed in § 3 of Chapter II~~, and the periodic contact problem, which comprises content of § 5 of Chapter II.

In § 8 of Chapter II an attempt is made to calculate surface deformations, which up till now were not calculated in the theory of the contact problem.

Chapter III gives a number of new solutions ~~of~~ an axisymmetric contact problem of the theory of elasticity.

In Chapter IV, together with the classical solutions, a number of new solutions belonging to the authors is given.

The book should be ~~examined~~ as a division of the mathematical theory of elasticity, since it is devoted to the solution of basic contact problems of the theory of elasticity.

Basic information on the theory of the contact problem can be found in courses of Academician L. S. Leybenzon[1] and S. P. Timoshenko[2].

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[1]Leybenzon L. S., Kurs teorii uprugosgi,(Course of the theory of elasticity), Gostekhizdat, 1947.

[2]Timoshenko S. P., Teoriya uprugosti (Theory of elasticity), ONTI, 1937.

CHAPTER I

MATHEMATICAL INTRODUCTION[1]

§ 1. Reduction of the Fundamental Equation of the Two-Dimensional Contact Problem to the Dirichlet Problem for a Circle

Let us gein from the consideration of the fundamental equation of the two-dimenstionsl contact problem of the theory of elasticity:

$$\int_{-a}^{a} p(t) \ln \frac{1}{|x-t|} dt = f(x), \quad -a < x < a, \tag{1}$$

where $f(x)$ is function assigned inside the interval $(-a, a)$, $p(t)$ is the unknown function, which must be determined, inside interval $(-a, a)$ in such a manner so that equation (1) is satisfied. Relative to the assigned function $f(x)$ we will assume that it is continuous, and a derivative of it $f'(x)$ can have points of discontinuity inside the interval $(-a, a)$.

Let us consider the function of two variable

$$V(x, y) = \int_{-a}^{a} p(t) \ln \frac{1}{R} dt, \tag{2}$$

where

$$R = \sqrt{(x-t)^2 + y^2}. \tag{3}$$

When $y = 0$ R' turns into $|x-t|$ and function $V(x, y)$ turns into the left site of equation (1). Thus, equation (1) is equivalent to the

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[1] In this chapter we give in detail and as elementary as possible the discussed solutions of certain equations on which the theory of the contact problems, placed in Chapter II is based.

condition

$$V(x, 0) = f(x), \quad -a < x < a, \tag{4}$$

imposed on function $V(x, y)$. Function $V(x, y)$, defined by relation (2), is called the logarithmic potential of the simple layer on the segment of the Ox $-a < x < a$ with density $p(t)$. Solution of the initial equation (1) is equivalent to the detecting of the density of the simple layer, the logarithmic potential of which $V(x, y)$ turns into the assigned function $f(x)$ on the segment of $-a < x < a$. Before turning to the solution of this problem, let us investigate in greater detail properties of the potential of the simple layer $V(x, y)$. If the point with coordinates x, y does not lie on the segment of Ox $-a < x < a$, partial derivatives of function $V(x, y)$ can be calculated by direct differentiation under the integral sign in the right side of relation (2).

Consecutively we find:

$$\ln\frac{1}{R} = -\frac{1}{2}\ln R^2 = -\frac{1}{2}\ln[(x-t)^2+y^2],$$

$$\frac{\partial}{\partial x}\ln\frac{1}{R} = -\frac{x-t}{(x-t)^2+y^2}, \quad \frac{\partial}{\partial y}\ln\frac{1}{R} = -\frac{y}{(x-t)^2+y^2},$$

$$\frac{\partial^2}{\partial x^2}\ln\frac{1}{R} = \frac{(x-t)^2-y^2}{[(x-t)^2+y^2]^2}, \quad \frac{\partial^2}{\partial y^2}\ln\frac{1}{R} = \frac{-(x-t)^2+y^2}{[(x-t)^2+y^2]^2}.$$

Hence

$$\frac{\partial V(x, y)}{\partial x} = -\int_{-a}^{a}\frac{p(t)(x-t)\,dt}{(x-t)^2+y^2}, \quad \frac{\partial V(x, y)}{\partial y} = -\int_{-a}^{a}\frac{p(t)\,y\,dt}{(x-t)^2+y^2},$$

$$\frac{\partial^2 V(x, y)}{\partial x^2} = \int_{-a}^{a}\frac{p(t)[(x-t)^2-y^2]\,dt}{[(x-t)^2+y^2]^2}, \tag{5}$$

$$\frac{\partial^2 V(x, y)}{\partial y^2} = -\int_{-a}^{a}\frac{p(t)[(x-t)^2-y^2]\,dt}{[(x-t)^2+y^2]^2}.$$

From relation (5) it is obvious that function $V(x, y)$ satisfies the partial differential equation

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0. \tag{6}$$

Differential equation (6) is called the Laplace equation: any function satisfying the Laplace equation is called the harmonic function. Thus, function $V(x, y)$ everywhere in plane xOy, with the exception of points of the segment of $-a<x<a$, of the Ox axis is of a harmonic function.

Let us investigate now the behavior of the partial derivative $\frac{\partial V}{\partial y}$ with the approach of the point with coordinates x, y to the segment of the Ox axis $-a<x<a$. When $-a<x<a$ the definite integral, which determines the derivative $\frac{\partial V}{\partial y}$ in formulas (5), can be divided into three integrals:

$$-\int_{-a}^{a}\frac{p(t)\,y\,dt}{(x-t)^2+y^2}=$$
$$=-y\int_{-a}^{x-\varepsilon}\frac{p(t)\,dt}{(x-t)^2+y^2}-y\int_{x-\varepsilon}^{x+\varepsilon}\frac{p(t)\,dt}{(x-t)^2+y^2}-y\int_{x+\varepsilon}^{a}\frac{p(t)\,dt}{(x-t)^2+y^2}. \tag{7}$$

The second of the definite integrals of the right side of formula (7) in turn can be represented in the form of the sum of two integrals:

$$-y\int_{x-\varepsilon}^{x+\varepsilon}\frac{p(t)\,dt}{(x-t)^2+y^2}=-yp(x)\int_{x-\varepsilon}^{x+\varepsilon}\frac{dt}{(x-t)^2+y^2}+y\int_{x-\varepsilon}^{x+\varepsilon}\frac{[p(x)-p(t)]\,dt}{(x-t)^2+y^2}, \tag{8}$$

if function $p(t)$ is continuous at point $t = x$.

Thus,

$$\frac{\partial V(x,y)}{\partial y}=J_1(x, y)+J_2(x, y)+J_3(x, y)+J_4(x, y), \tag{9}$$

where

$$J_1(x, y)=-y\int_{-a}^{x-\varepsilon}\frac{p(t)\,dt}{(x-t)^2+y^2},\quad J_2(x, y)=-y\int_{x+\varepsilon}^{a}\frac{p(t)\,dt}{(x-t)^2+y^2}, \tag{10}$$

$$J_3(x, y) = -yp(x) \int_{x-\delta}^{x+\delta} \frac{dt}{(x-t)^2+y^2}, \tag{11}$$

$$J_4(x, y) = y \int_{x-\delta}^{x+\delta} \frac{[p(x)-p(t)]\,dt}{(x-t)^2+y^2}. \tag{12}$$

Assuming in (10) $y = 0$, we will find

$$J_1(x, 0) = J_2(x, 0) = 0, \tag{13}$$

since in definite integrals

$$\int_{-\infty}^{x-\delta} \frac{p(t)\,dt}{(x-t)^2} \text{ and } \int_{x+\delta}^{\infty} \frac{p(t)\,dt}{(x-t)^2} \tag{14}$$

integrands are limited, and, consequently, the very definite integrals (14) are limited. Assuming

$$t = x - |y| \operatorname{tg} \alpha \quad \left(-\frac{\pi}{2} < \alpha < \frac{\pi}{2}\right),$$

we will find:

$$\int_{x-\delta}^{x+\delta} \frac{dt}{(x-t)^2+y^2} = \frac{1}{|y|} \int_{-\alpha_0}^{\alpha_0} d\alpha = \frac{2\alpha_0}{|y|}, \tag{15}$$

where

$$\alpha_0 = \operatorname{arctg} \frac{\delta}{|y|} \quad \left(0 < \alpha_0 < \frac{\pi}{2}\right). \tag{16}$$

Substituting (15) into (11), we will find

$$J_3(x, y) = -\frac{y}{|y|} 2\alpha_0 p(x),$$

or

$$\left.\begin{aligned} J_3(x, y) &= -2\alpha_0 p(x) \quad \text{when} \quad y > 0, \\ J_3(x, y) &= 2\alpha_0 p(x) \quad \text{when} \quad y < 0. \end{aligned}\right\} \tag{17}$$

As can be seen from (16),

$$\alpha_0 = \frac{\pi}{2} \text{ when } y = 0. \tag{18}$$

Thus, if coordinate y tends to zero, remaining positive, function $J_3(x, y)$ tends to the limiting value $-\pi p(x)$; if coordinate y approaches zero, remaining negative, function $J_3(x, y)$ approaches the value $\pi p(x)$. Function $J_3(x, y)$ undergoes discontinuity when $y = 0$,

$$J_3(x, +0) = -\pi p(x), \quad J_3(x, -0) = \pi p(x); \tag{19}$$

here $J_3(x, +0)$ and $J_3(x, -0)$ denote the limiting values of function $J_3(x, y)$ on various sides of the point of discontinuity.

Using relations (13) and (19), from formulas (9) we will find

$$\left.\begin{aligned} \frac{\partial V(x, +0)}{\partial y} + \pi p(x) &= J_4(x, +0), \\ \frac{\partial V(x, -0)}{\partial y} - \pi p(x) &= J_4(x, -0). \end{aligned}\right\} \tag{20}$$

From (12) we find

$$|J_4(x)| < \eta |y| \int_{x-\varepsilon}^{x+\varepsilon} \frac{dt}{(x-t)^2 + y^2}, \tag{21}$$

where η is the maximum absolute value of the difference $p(t) - p(x)$ when $x - \varepsilon \leqslant t \leqslant x + \varepsilon$. Since function $p(t)$ by assumption is continuous at point $t = x$, then η will be as small as desired at sufficiently small ε. Substituting (15) into (21) and taking into account (16), we will find

$$|J_4(x, y)| < 2\alpha_0 \eta \leqslant \pi \eta. \tag{22}$$

On the basis (22) from (20) it follows that

$$\left.\begin{array}{l}\left|\frac{\partial V(x,+0)}{\partial y}+\pi p(x)\right|\leqslant\pi\eta,\\ \left|\frac{\partial V(x,-0)}{\partial y}-\pi p(x)\right|\leqslant\pi\eta.\end{array}\right\} \tag{23}$$

Since inequalities (23) are accurate at any ε as small as desired, and η approaches zero together with ε, then

$$\left.\begin{array}{l}\frac{\partial V(x,+0)}{\partial y}=-\pi p(x),\\ \frac{\partial V(x,-0)}{\partial y}=\pi p(x),\end{array}\right\} \tag{24}$$

if at point $t = x$ function $p(x)$, does not undergo discontinuity.

Let us investigate now the behavior of function $V(x, y)$ with removal at infinity of the point with coordinates x, y. Assuming in (3)

$$x=r\cos\varphi,\quad y=r\sin\varphi,$$

we will find

$$\begin{gathered}R=\sqrt{r^2-2rt\cos\varphi+t^2}=r\sqrt{1-2\frac{t}{r}\cos\varphi+\frac{t^2}{r^2}},\\ \ln\frac{1}{R}=\ln\frac{1}{r}-\ln\sqrt{1-2\frac{t}{r}\cos\varphi+\frac{t^2}{r^2}}.\end{gathered} \tag{25}$$

Substituting (25) into (2), we will find

$$\begin{gathered}V(x,y)-\ln\frac{1}{r}\int_{-a}^{a}p(t)\,dt=\\ =-\int_{-a}^{a}p(t)\ln\sqrt{1-2\frac{t}{r}\cos\varphi+\frac{t^2}{r^2}}\,dt,\end{gathered} \tag{26}$$

where the right side of equality (26) approaches zero when $r \to \infty$. Introducing the designation

$$\int_{-\infty}^{\infty} p(\xi)\, d\xi = P, \tag{27}$$

we will find

$$V(x, y) - P \ln \frac{1}{r} \to 0, \quad \text{whence} \quad r \to \infty \quad (r = \sqrt{x^2 + y^2}). \tag{28}$$

Thus, the solution of the initial equation (1) is reduced by us to the construction of function $V(x, y)$, harmonic in the whole plane x, y, except points of the segment of the axis Ox $-a < x < a$, and satisfying conditions (4) and (28)[1]. Having constructed function $V(x, y)$, we will find the unknown function $p(x)$ according to one of the formulas (24).

Before turning to the construction of function $V(x, y)$, let us show one important property which the Laplace equation (6) possesses. Let us produce in this equation the change in variables x, y by variables ε, η, having assumed that

$$\left.\begin{aligned} x &= x(\xi, \eta), \\ y &= y(\xi, \eta). \end{aligned}\right\} \tag{29}$$

Consecutively we find

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[1]The problem of construction of the harmonic function according to boundary values assigned to it in the theory of potential is called the Dirichlet problem. The existence and uniqueness of the solution of this problem are probem with very general assumptions. In our book these investigations are not discussed.

$$\frac{\partial V}{\partial \xi}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial \xi}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial \xi}, \tag{30}$$

$$\frac{\partial V}{\partial \eta}=\frac{\partial V}{\partial x}\frac{\partial x}{\partial \eta}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial \eta}, \tag{31}$$

$$\frac{\partial^2 V}{\partial \xi^2}=\frac{\partial^2 V}{\partial x^2}\left(\frac{\partial x}{\partial \xi}\right)^2+2\frac{\partial^2 V}{\partial x\,\partial y}\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi}+\frac{\partial^2 V}{\partial y^2}\left(\frac{\partial y}{\partial \xi}\right)^2+\frac{\partial V}{\partial x}\frac{\partial^2 x}{\partial \xi^2}+\frac{\partial V}{\partial y}\frac{\partial^2 y}{\partial \xi^2}, \tag{32}$$

$$\frac{\partial^2 V}{\partial \eta^2}=\frac{\partial^2 V}{\partial x^2}\left(\frac{\partial x}{\partial \eta}\right)^2+2\frac{\partial^2 V}{\partial x\,\partial y}\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta}+\frac{\partial^2 V}{\partial y^2}\left(\frac{\partial y}{\partial \eta}\right)^2+\frac{\partial V}{\partial x}\frac{\partial^2 x}{\partial \eta^2}+\frac{\partial V}{\partial y}\frac{\partial^2 y}{\partial \eta^2}, \tag{33}$$

whence

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}=\frac{\partial^2 V}{\partial x^2}\left[\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial x}{\partial \eta}\right)^2\right]+2\frac{\partial^2 V}{\partial x\,\partial y}\left(\frac{\partial x}{\partial \xi}\frac{\partial y}{\partial \xi}+\frac{\partial x}{\partial \eta}\frac{\partial y}{\partial \eta}\right)+$$
$$+\frac{\partial^2 V}{\partial y^2}\left[\left(\frac{\partial y}{\partial \xi}\right)^2+\left(\frac{\partial y}{\partial \eta}\right)^2\right]+\frac{\partial V}{\partial x}\left(\frac{\partial^2 x}{\partial \xi^2}+\frac{\partial^2 x}{\partial \eta^2}\right)+\frac{\partial V}{\partial y}\left(\frac{\partial^2 y}{\partial \xi^2}+\frac{\partial^2 y}{\partial \eta^2}\right). \tag{34}$$

Let us now connect function $y(\xi, \eta)$ with function $x(\xi, \eta)$ by relations

$$\left.\begin{aligned}\frac{\partial y}{\partial \xi}&=-\frac{\partial x}{\partial \eta},\\ \frac{\partial y}{\partial \eta}&=\frac{\partial x}{\partial \xi}.\end{aligned}\right\} \tag{35}$$

The necessary and sufficient condition of the existence of function $y(\xi, \eta)$, which satisfies relations (35), is the condition

$$\frac{\partial}{\partial \eta}\left(-\frac{\partial x}{\partial \eta}\right)=\frac{\partial}{\partial \xi}\left(\frac{\partial x}{\partial \xi}\right),$$

i.e., condition

$$\frac{\partial^2 x}{\partial \xi^2}+\frac{\partial^2 x}{\partial \eta^2}=0. \tag{36}$$

If condition (36) is fulfilled, then function $y(\xi, \eta)$ can be found by function $x(\xi, \eta)$ from relation (35) by means of quadratures. Here function $y(\xi, \eta)$ will be determined with an accuracy of the arbitrary constant term. From relations (35) it follows that the thus found $y(\xi, \eta)$ will satisfy equation

$$\frac{\partial^2 y}{\partial \xi^2}+\frac{\partial^2 y}{\partial \eta^2}=0. \tag{37}$$

Thus, if two functions $x(\xi, \eta)$ and $y(\xi, \eta)$ satisfy conditions (35), they must be harmonic functions, as one can see from (36) and (37). If harmonic function $x(\xi, \eta)$ is assigned, then harmonic function $y(\xi, \eta)$, connected with it by conditions (35) can be found by means of quadratures with an accuracy up to the arbitrary constant component. Under conditions (35) the harmonic function $y(\xi, \eta)$ is called the function conjugate with the harmonic function $x(\xi, \eta)$.

Substituting (35) into (34) and taking into account (36) and (37), we will find that under conditions (35) the relation will take place

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}=\left(\frac{\partial^2 V}{\partial x^2}+\frac{\partial^2 V}{\partial y^2}\right)\left[\left(\frac{\partial x}{\partial \xi}\right)^2+\left(\frac{\partial x}{\partial \eta}\right)^2\right]. \tag{38}$$

This relation shows that if function $V(x, y)$ satisfies the Laplace equation (6), then after a change in variables (29) this function will satisfy equation

$$\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}=0, \tag{39}$$

i.e., the Laplace equation retains its form with a change in variables (6) if conditions (35) are fulfilled. In other words, if in the expression for the harmonic function $V(x, y)$ a change in variables x, y is produced by variables ξ, η, then we will again obtain the harmonic function of new variables ξ, η under conditions (35). This property of harmonic functions is widely used in the solution of the boundary value problems. Actually, if it is required to construct function $V(x, y)$, which is harmonic in a certain region g and satisfies the assigned boundary conditions on the boundary of this region, then by producing a change in variables x, y by variables ξ, η, we will arrive at the problem of construction of the harmonic function of new variables ξ, η according to boundary conditions assigned already on the boundary of the new g*, into which region g passes as a result of the transformation of the variables. In particular, if one were to find function $x(\xi, \eta)$ and $y(\xi, \eta)$, which satisfy conditions (35) and transfer region g in plane xOy into a circle $\xi^2+\eta^2<1$ in plane $\xi O\eta$, then it is possible to reduce the construction

of the harmonic function according to boundary conditions assigned on the boundary of region g to the construction of the harmonic function according to boundary conditions assigned on the circle.

By examining the solution of the initial equation (1), we arrived at the construction of function $V(x, y)$, harmonic in the whole plane xOy with the exception of the segment of the Ox $-a<x<a$, according to the boundary condition (4) and subsidiary condition (28). Let us show that the transformation of variables x, y into variables ξ, η

$$\left.\begin{aligned} x&=\frac{a\xi}{2}\left(1+\frac{1}{\xi^2+\eta^2}\right),\\ y&=\frac{a\eta}{2}\left(1-\frac{1}{\xi^2+\eta^2}\right),\end{aligned}\right\} \qquad (40)$$

satisfies conditions (35) and turns the whole plane xOy, with the exception of the segment of the Ox axis $-a<x<a$, into the circle $\xi^2+\eta^2<1$ on plane $\xi O\eta$ (solving (40) relative to ξ and η, we will obtain two real solutions; from these solutions below we will take for which $\xi^2+\eta^2<1$):

Differentiating (40), we will find

$$\left.\begin{aligned} &\frac{\partial x}{\partial\xi}=\frac{a}{2}\left[1+\frac{\eta^2-\xi^2}{(\xi^2+\eta^2)^2}\right], && \frac{\partial x}{\partial\eta}=-\frac{a\xi\eta}{(\xi^2+\eta^2)^2},\\ &\frac{\partial y}{\partial\xi}=\frac{a\xi\eta}{(\xi^2+\eta^2)^2}, && \frac{\partial y}{\partial\eta}=\frac{a}{2}\left[1+\frac{\eta^2-\xi^2}{(\xi^2+\eta^2)^2}\right].\end{aligned}\right\} \qquad (41)$$

Thus, conditions (35) are satisfied. Assuming in (40)

$$\xi=\rho\cos\vartheta,\quad \eta=\rho\sin\vartheta, \qquad (42)$$

i.e., passing to polar coordinates ρ, ϑ on plane $\xi O\eta$, we will find

$$\left.\begin{aligned} x&=\frac{a}{2}\left(\rho+\frac{1}{\rho}\right)\cos\vartheta,\\ y&=\frac{a}{2}\left(\rho-\frac{1}{\rho}\right)\sin\vartheta,\end{aligned}\right\} \qquad (43)$$

whence

$$\frac{x^2}{\left[\frac{a}{2}\left(\frac{1}{\rho}+\rho\right)\right]^2}+\frac{y^2}{\left[\frac{a}{2}\left(\frac{1}{\rho}-\rho\right)\right]^2}=1. \qquad (44)$$

Equation (44) shows that points of the plane $\xi O\eta$, lying on the circle of radius ρ:

$$\xi^2+\eta^2=\rho^2 \quad (\rho<1), \qquad (45)$$

correspond the point of plane xOy lying on the ellipse (44) with semiaxes $\frac{a}{2}\left(\frac{1}{\rho}+\rho\right)$ and $\frac{a}{2}\left(\frac{1}{\rho}-\rho\right)$. When the radius of the circle ρ approaches zero, the semiaxes of the ellipse increase without limit, point $\rho = 0$ in plane $\xi O\eta$ corresponds to the point at infinity of plane xOy When ρ approaches unity, the semimajor axis of the ellipse $\frac{a}{2}\left(\frac{1}{\rho}+\rho\right)$ approaches a, the semiminor axis of the ellipse $\frac{a}{2}\left(\frac{1}{\rho}-\rho\right)$ approaches zero, and circle $\rho = 1$ on plane $\xi O\eta$ corresponds to the segment of the axis Ox $-a<x<a$. From (43) it is clear that

$$\left.\begin{array}{l} y\leqslant 0 \text{ when } 0\leqslant\vartheta\leqslant\pi, \\ y\geqslant 0 \text{ when } \pi\leqslant\vartheta\leqslant 2\pi. \end{array}\right\} \qquad (46)$$

Thus, the upper semicircle $\rho = 1$ on plane $\xi O\eta$ corresponds to the lower side of the segment of the axis Ox $-a<x<a$; and the lower semicircumference – to the upper side of this segment. Assuming $\rho = 1$ in the first of relations (43), we will obtain the dependence

$$x=a\cos\vartheta, \qquad (47)$$

which connects the position of the point on the segment of the Ox axis $-a<x<a$ with the position of the point corresponding to it on the circle $\rho = 1$ in plane $\xi O\eta$.

Thus, if in the expression for the unknown function $V(x, y)$, harmonic outside the segment of the Ox axis $-a<x<a$, one replaces variables x, y by variables ξ, η according to (40), then we will obtain the function of variables ξ, η, harmonic inside the

circumference $\xi^2+\eta^2=1$, i.e., satisfying inside this circumference the Laplace equation (39) in variables ξ, η. Boundary condition (4), according to (47), will take the form:

$$V = f(a\cos\vartheta) \quad \text{when } \rho = 1. \tag{48}$$

From (39) we find

$$r^2 = x^2+y^2 = \frac{a^2}{4}\left[\left(\rho+\frac{1}{\rho}\right)^2\cos^2\vartheta+\left(\rho-\frac{1}{\rho}\right)^2\sin^2\vartheta\right] =$$
$$= \frac{a^2}{4}\left(\rho^2+2\cos 2\vartheta+\frac{1}{\rho^2}\right) = \frac{a^2}{4\rho^2}(\rho^4+2\rho^2\cos 2\vartheta+1),$$

whence

$$\frac{1}{r} = \frac{2\rho}{a\sqrt{\rho^4+2\rho^2\cos 2\vartheta+1}}. \tag{49}$$

Taking the Logarithm of (49), we will find

$$\ln\frac{1}{r}-\ln\frac{2\rho}{a} = -\ln\sqrt{\rho^4+2\rho^2\cos 2\vartheta+1}. \tag{50}$$

Hence

$$\ln\frac{1}{r}-\ln\frac{2\rho}{a}\to 0 \quad \text{when } \rho\to 0,\ r\to\infty. \tag{51}$$

On the basis of (51) we find that condition (28) will now take the form

$$V-P\ln\frac{2\rho}{a}\to 0 \quad \text{when } \rho\to 0. \tag{52}$$

Thus, the solution of the initial equation (1) was reduced to the construction of function V, which satisfies the Laplace equation (39) inside the circumference $\rho=1$ $(\rho=\sqrt{\xi^2+\eta^2})$ and conditions (48) and (52) on the circumference $\rho = 1$ and at the beginning of coordinates $\rho = 0$. Using the relations (43), we will find:

$$\frac{\partial V}{\partial\rho} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial\rho}+\frac{\partial V}{\partial y}\frac{\partial y}{\partial\rho} =$$
$$= \frac{a}{2}\left[\frac{\partial V}{\partial x}\left(1-\frac{1}{\rho^2}\right)\cos\vartheta+\frac{\partial V}{\partial y}\left(1+\frac{1}{\rho^2}\right)\sin\vartheta\right], \tag{53}$$

whence

$$\lim_{\rho\to 1}\left(\frac{\partial V}{\partial \rho}\right)=a\sin\vartheta \lim_{y\to 0}\left(\frac{\partial V}{\partial y}\right), \tag{54}$$

since $y \to 0$, when $\rho \to 1$. But, as one can see from (46), when $0<\vartheta<\pi$ $y\to -0$ (i.e., approaches zero, remaining negative) when $\rho \to 1$, and when $\pi<\vartheta<2\pi$ $y\to +0$, when $\rho \to 1$. Thus, according to (54)

$$\left.\begin{aligned}\left(\frac{\partial V}{\partial \rho}\right)_{\rho=1}&=a\sin\vartheta\left(\frac{\partial V}{\partial y}\right)_{y=-0} \text{ when } 0<\vartheta<\pi,\\ \left(\frac{\partial V}{\partial \rho}\right)_{\rho=1}&=a\sin\vartheta\left(\frac{\partial V}{\partial y}\right)_{y=+0} \text{ when } \pi<\vartheta<2\pi.\end{aligned}\right\} \tag{55}$$

Substituting (24) into (55), we will find

$$\left.\begin{aligned}\left(\frac{\partial V}{\partial \rho}\right)_{\rho=1}&=\pi a\sin\vartheta\, p(a\cos\vartheta) \quad \text{when } 0<\vartheta<\pi,\\ \left(\frac{\partial V}{\partial \rho}\right)_{\rho=1}&=-\pi a\sin\vartheta\, p(a\cos\vartheta) \text{ when } \pi<\vartheta<2\pi\end{aligned}\right\} \tag{56}$$

(we replaced argument x of function $\rho(x)$ by $a\cos\vartheta$ according to (47)). Relations (56) can be given the form

$$\left(\frac{\partial V}{\partial \rho}\right)_{\rho=1}=\pi a|\sin\vartheta|\,p(a\cos\vartheta)' \quad (0<\vartheta<2\pi). \tag{57}$$

Having constructed the unknown function V, which satisfies when $\rho = 1$ the Laplace equation (39) and satisfies conditions (48) and (52) when $\rho = 1$ and when $\rho = 0$, we will find the unknown function ρ from the relation (57).

Let us turn in the Laplace equation (39) from rectangular coordinates ξ, η to polar coordinates ρ, ϑ, assuming

$$\xi=\rho\cos\vartheta, \quad \eta=\rho\sin\vartheta.$$

Let us find

$$\frac{\partial V}{\partial \rho}=\frac{\partial V}{\partial \xi}\cos\theta+\frac{\partial V}{\partial \eta}\sin\theta,$$

$$\frac{\partial^2 V}{\partial \rho^2}=\frac{\partial^2 V}{\partial \xi^2}\cos^2\theta+2\frac{\partial^2 V}{\partial \xi\,\partial \eta}\sin\theta\cos\theta+\frac{\partial^2 V}{\partial \eta^2}\sin^2\theta,$$

$$\frac{\partial V}{\partial \theta}=-\frac{\partial V}{\partial \xi}\rho\sin\theta+\frac{\partial V}{\partial \eta}\rho\cos\theta,$$

$$\frac{\partial^2 V}{\partial \theta^2}=\frac{\partial^2 V}{\partial \xi^2}\rho^2\sin^2\theta-2\frac{\partial^2 V}{\partial \xi\,\partial \eta}\rho^2\sin\theta\cos\theta+\frac{\partial^2 V}{\partial \eta^2}\rho^2\cos^2\theta-$$
$$-\frac{\partial V}{\partial \xi}\rho\cos\theta-\frac{\partial V}{\partial \eta}\rho\sin\theta.$$

Hence

$$\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}=\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}. \qquad (58)$$

Thus, if in rectangular coordinates ξ, η function V satisfies the differential equation (39), then in polar coordinates ρ, θ this function V satisfies the differential equation

$$\frac{\partial^2 V}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial V}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 V}{\partial \theta^2}=0. \qquad (59)$$

Let us now examine function

$$W(\rho,\theta)=V-P\ln\frac{2\rho}{\epsilon}. \qquad (60)$$

Substituting V from (60) into (59), we will find

$$\frac{\partial^2 W}{\partial \rho^2}+\frac{1}{\rho}\frac{\partial W}{\partial \rho}+\frac{1}{\rho^2}\frac{\partial^2 W}{\partial \theta^2}=0, \qquad (61)$$

i.e., function $W(\rho,\theta)$, defined by relation (60), will also satisfy the Laplace equation in polar coordinates. On the basis (48) and (52) we will find

$$W(1,\vartheta)=f(a\cos\vartheta)-P\ln\frac{2}{a}, \tag{62}$$

$$W(0,\vartheta)=0. \tag{63}$$

Substituting V from (60) into (57), we will find

$$\frac{\partial W(1,\vartheta)}{\partial\rho}+P=\pi a|\sin\vartheta|\,p(a\cos\vartheta),$$

whence

$$p(a\cos\vartheta)=\frac{P+\frac{\partial W(1,\vartheta)}{\partial\rho}}{\pi a|\sin\vartheta|}. \tag{64}$$

Thus, having discovered function $W(\rho,\vartheta)$, which satisfies the differential equation (61) when $\rho < 1$ conditions (62) and (63) when $\rho = 1$ and $\rho = 0$, we will find the unknown function p from the relation (64). The construction of the function, harmonic inside the given circumference and taking the rated values on this circumference, is the subject of the following paragraph.

§ 2. Certain Methods of Resolution of the Dirichlet problem for a Circle

Let us expand function $W(1, \vartheta)$ of argument ϑ, defined by relation (62), in Fourier series. Since function $W(1,\vartheta)$, according to (62), is even, i.e., $W(1,-\vartheta)=W(1,\vartheta)$, this series will contain only cosines of angles multiple of ϑ, i.e., it will have the form

$$W(1,\vartheta)=f(a\cos\vartheta)-P\ln\frac{2}{a}=\alpha_0+\sum_{n=1}^{\infty}\alpha_n\cos n\vartheta. \tag{65}$$

Coefficients of this Fourier series $\alpha_0, \alpha_1, \alpha_2, \ldots$ can, as is known, be found by means of quadratures by formulas

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} W(1,\vartheta)\,d\vartheta,$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} W(1,\vartheta)\cos n\vartheta\,d\vartheta, \quad n = 1, 2, \ldots,$$

or

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(a\cos\vartheta)\,d\vartheta - P\ln\frac{2}{a},$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(a\cos\vartheta)\cos n\vartheta\,d\vartheta, \quad n = 1, 2, \ldots \tag{66}$$

Let us now show that function $W(\rho,\vartheta)$, which satisifes the differential equation (61) when $\rho < 1$ and condition (65) when $\rho = 1$, can be found in the form of the series

$$W(\rho,\vartheta) = a_0 + \sum_{n=1}^{\infty} a_n \rho^n \cos n\vartheta. \tag{67}$$

Let us find

$$\frac{\partial W}{\partial \rho} = \sum_{n=1}^{\infty} a_n n \rho^{n-1}\cos n\vartheta, \quad \frac{\partial^2 W}{\partial \rho^2} = \sum_{n=1}^{\infty} a_n n(n-1)\rho^{n-2}\cos n\vartheta,$$

$$\frac{\partial^2 W}{\partial \vartheta^2} = -\sum_{n=1}^{\infty} a_n n^2 \rho^n \cos n\vartheta. \tag{68}$$

Substituting (68) into (61), we will find

$$\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \vartheta^2} = \sum_{n=1}^{\infty} a_n\left[n(n-1) + n - n^2\right]\rho^{n-2}\cos n\vartheta = 0,$$

i.e., function $W(\rho,\vartheta)$, defined by series (67), indeed satisfies the differential equation (61).

Assuming in (67) $\rho = 1$, we will be convinced in the fact that condition (65) is fulfilled.

Substituting (67) into (63), we will find that equality should take place

$$\alpha_0 = 0. \tag{69}$$

Condition (69) determines the constant P, which up till now has remained indefinite. Actually, by assuming α_0 from (66) into (69), we will find

$$P = \frac{1}{2\pi \ln \frac{2}{a}} \int_0^{2\pi} f(a\cos\vartheta)\, d\vartheta. \tag{70}$$

Substituting (67) into (64), we will find

$$p(a\cos\vartheta) = \frac{P + \sum_{n=1}^{\infty} \alpha_n n \cos n\vartheta}{\pi a |\sin\vartheta|}. \tag{71}$$

We will not touch upon the question about the convergence of series formally obtained by us (71).

Substituting $P, \alpha_1, \alpha_2, \ldots$ from (70) and (66) into (71) and replacing in (71) $a\cos\vartheta$ by x, we will find the unknown solution $p(x)$ of the initial equation (1).

Let us examine the examples.

$$1) \quad f(x) = \alpha = \text{const}. \tag{72}$$

As can be seen from (66), in this case

$$\alpha_1 = \alpha_2 = \ldots = 0.$$

From (70) we find

$$P = \frac{\alpha}{\ln \frac{2}{a}}. \tag{73}$$

Formula (71) gives:

$$p(a\cos\vartheta)=\frac{P}{\pi a|\sin\vartheta|}.$$

Assuming $a\cos\vartheta=x$, we will find

$$a|\sin\vartheta|=\sqrt{a^2-a^2\cos^2\vartheta}=\sqrt{a^2-x^2},$$

$$p(x)=\frac{P}{\pi\sqrt{a^2-x^2}}. \tag{74}$$

Formulas (74) and (73) give for the given example the solution of the initial equation (1).

$$2)\quad f(x)=\alpha-Ax^2. \tag{75}$$

In this case

$$f(a\cos\vartheta)=\alpha-Aa^2\cos^2\vartheta=\alpha-\frac{1}{2}Aa^2-\frac{1}{2}Aa^2\cos 2\vartheta,$$

whence, on the basis of (66) and (70),

$$\alpha_1=0,\quad \alpha_2=-\frac{1}{2}Aa^2,\quad \alpha_3=\alpha_4=\ldots=0,$$

$$P=\frac{\alpha-\frac{1}{2}Aa^2}{\ln\frac{2}{a}}. \tag{76}$$

From (71) we find

$$p(a\cos\vartheta)=\frac{1}{\pi a|\sin\vartheta|}(P-Aa^2\cos 2\vartheta)=$$
$$=\frac{1}{\pi a|\sin\vartheta|}(P+Aa^2-2Aa^2\cos^2\vartheta).$$

Assuming in this relation $a\cos\vartheta=x$, we will find

$$p(x)=\frac{P+Aa^2-2Ax^2}{\pi\sqrt{a^2-x^2}}. \tag{77}$$

Formulas (77) and (76) give the solution of the initial equation (1) for the case when the right side of this equation $f(x)$ has the form of (75).

In general the formula (71) established by us gives the solution of the initial equation (1) in the form of an infinite series.

Let us now show another procedure of resolution of the problem, which made possible in general to obtain the solution of the initial equation (1) in the form of a definite integral. Let us prove that function $W(\rho, \vartheta)$, which satisfies the Laplace equation (61) with $\rho < 1$ and takes rated values with $\rho = 1$

$$W(1, \vartheta) = F(\vartheta), \tag{78}$$

can be represented by formula

$$W(\rho, \vartheta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{F(\varphi)(1-\rho^2)\, d\varphi}{1 - 2\rho\cos(\varphi - \vartheta) + \rho^2}. \tag{79}$$

The right side of formula (79) is called the Poisson integral. With $\rho < 1$ partial derivatives of this integral with respect to ρ and ϑ can be found by direct differentiation under the sign of the integral. Let us find

$$\frac{\partial W}{\partial \rho} = \frac{1}{\pi} \int_0^{2\pi} \frac{F(\varphi)\,[(1+\rho^2)\cos(\varphi-\vartheta) - 2\rho]\, d\varphi}{[1 - 2\rho\cos(\varphi-\vartheta) + \rho^2]^2},$$

$$\frac{\partial^2 W}{\partial \rho^2} = \frac{2}{\pi} \int_0^{2\pi} \frac{F(\varphi)\,[2\cos^3(\varphi-\vartheta) - \rho(3+\rho^2)\cos^2(\varphi-\vartheta) - 1 + 3\rho^2]\, d\varphi}{[1 - 2\rho\cos(\varphi-\vartheta) + \rho^2]^3},$$

$$\frac{\partial^2 W}{\partial \vartheta^2} = -\frac{1}{\pi} \int_0^{2\pi} \frac{F(\varphi)(1-\rho^2)\,\rho\,[2\rho\cos^2(\varphi-\vartheta) + (1+\rho^2)\cos(\varphi-\vartheta) - 4\rho]\, d\varphi}{[1 - 2\rho\cos(\varphi-\vartheta) + \rho^2]^3}.$$

Hence:

$$\begin{aligned}\frac{\partial^2 W}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial W}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2 W}{\partial \vartheta^2} = \frac{1}{\pi\rho}\int_0^{2\pi} \frac{F(\varphi)}{[1 - 2\rho\cos(\varphi-\vartheta) + \rho^2]^3} \{[4\rho - 2\rho(1+\rho^2) -& \\ - 2\rho(1-\rho^2)]\cos^2(\varphi-\vartheta) +& \\ + [-2\rho^2(3+\rho^2) + (1+\rho^2)^2 + 4\rho^2 - (1-\rho^2)(1+\rho^2)]\cos(\varphi-\vartheta) +& \\ + 2\rho(-1+3\rho^2) - 2\rho(1+\rho^2) + 4\rho(1-\rho^2)\}\, d\varphi = 0,&\end{aligned}$$

i.e., function $W(\rho,\theta)$, defined by relation (79), indeed satisfies differential equation (61) when $\rho < 1$. Let us now find the limit to which the integral of Poisson approaches when ρ approaches its limit value – unity. Having divided this integral into a sum of four integrals, we will give to formula (79) the form

$$W(\rho,\theta)=J_1(\rho,\theta)+J_2(\rho,\theta)+J_3(\rho,\theta)+J_4(\rho,\theta), \tag{79'}$$

where

$$J_1(\rho,\theta)=\frac{1}{2\pi}\int\limits_0^{\theta-\varepsilon}\frac{F(\varphi)(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2}, \tag{80}$$

$$J_2(\rho,\theta)=\frac{1}{2\pi}\int\limits_{\theta+\varepsilon}^{2\pi}\frac{F(\varphi)(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2}, \tag{81}$$

$$J_3(\rho,\theta)=\frac{F(\theta)}{2\pi}\int\limits_{\theta-\varepsilon}^{\theta+\varepsilon}\frac{(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2}, \tag{82}$$

$$J_4(\rho,\theta)=\frac{1}{2\pi}\int\limits_{\theta-\varepsilon}^{\theta+\varepsilon}\frac{[F(\varphi)-F(\theta)](1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2}. \tag{83}$$

Assuming in (80) and (81) $\rho = 1$, let us find

$$J_1(1,\theta)=J_2(1,\theta)=0, \tag{84}$$

since when $\rho = 1$ the integrands in definite integrals $J_1(\rho,\theta)$ and $J_2(\rho,\theta)$ turn into zero everywhere in the region of integration.

Assuming further

$$\operatorname{tg}\frac{\varphi-\theta}{2}=\frac{1-\rho}{1+\rho}\operatorname{tg}\alpha \quad \left(-\frac{\pi}{2}\leqslant\alpha\leqslant\frac{\pi}{2}\right),$$

let us find

$$\cos(\varphi-\theta)=\frac{1-\operatorname{tg}^2\frac{\varphi-\theta}{2}}{1+\operatorname{tg}^2\frac{\varphi-\theta}{2}}=\frac{(1+\rho)^2\cos^2\alpha-(1-\rho)^2\sin^2\alpha}{(1+\rho)^2\cos^2\alpha+(1-\rho)^2\sin^2\alpha}=$$

$$=\frac{2\rho+(1+\rho^2)\cos 2\alpha}{1+\rho^2+2\rho\cos 2\alpha},$$

$$1-2\rho\cos(\varphi-\theta)+\rho^2=1+\rho^2-2\rho\,\frac{2\rho+(1+\rho^2)\cos 2\alpha}{1+\rho^2+2\rho\cos 2\alpha}=$$

$$=\frac{(1-\rho^2)^2}{1+2\rho\cos 2\alpha+\rho^2},$$

$$\left(1+\operatorname{tg}^2\frac{\varphi-\theta}{2}\right)\frac{d\varphi}{2}=\frac{1-\rho}{1+\rho}\sec^2\alpha\,d\alpha,$$

$$d\varphi=2\,\frac{1-\rho}{1+\rho}\,\frac{\sec^2\alpha\,d\alpha}{1+\left(\frac{1-\rho}{1+\rho}\right)^2\operatorname{tg}^2\alpha}=\frac{2(1-\rho^2)\,d\alpha}{(1+\rho)^2\cos^2\alpha+(1-\rho)^2\sin^2\alpha}=$$

$$=\frac{2(1-\rho^2)\,d\alpha}{1+2\rho\cos 2\alpha+\rho^2},$$

and thus,

$$\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} \frac{(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\vartheta)+\rho^2} = 2\int_{-\alpha_0}^{\alpha_0} d\alpha = 4\alpha_0, \tag{85}$$

where

$$\alpha_0 = \operatorname{arctg}\frac{1+\rho}{1-\rho}\operatorname{tg}\frac{\varepsilon}{2} \quad \left(0<\alpha_0<\frac{\pi}{2}\right). \tag{86}$$

As can be seen from (85) and (86),

$$\alpha_0<\frac{\pi}{2} \text{ when } \rho<1, \quad \alpha_0=\frac{\pi}{2} \text{ when } \rho=1,$$

$$\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} \frac{(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\vartheta)+\rho^2} < 2\pi \text{ when } \rho<1, \tag{87}$$

$$\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} \frac{(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\vartheta)+\rho^2} = 2\pi \text{ when } \rho=1. \tag{88}$$

Substituting (88) into (82), let us find

$$J_2(1,\vartheta)=F(\vartheta). \tag{89}$$

Substituting (84) and (89) into (79'), let us find

$$W(1,\vartheta)-F(\vartheta)=J_1(1,\vartheta). \tag{90}$$

From (83) we find

$$\left|J_1(\rho,\vartheta)\right| < \frac{\eta}{2\pi}\int_{\vartheta-\varepsilon}^{\vartheta+\varepsilon} \frac{(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\vartheta)+\rho^2} < \eta \text{ when } \rho<1, \tag{91}$$

on the basis (87) and (88). Here η is the maximum absolute value of the difference $F(\varphi)-F(\vartheta)$ when $\vartheta-\varepsilon<\varphi<\vartheta+\varepsilon$. Since by assumption function $F(\varphi)$ is continuous at point $\varphi=\vartheta$, then η will be as small as desired when sufficiently small ε. From (90) and (91) there follows the inequality

$$|W(1,\vartheta)-F(\vartheta)|<\eta. \tag{92}$$

But since inequality (92) is accurate at any ε as small as desired,

and η approaches zero together with ε, from (92) it follows that

$$W(1, \theta) = F(\theta), \tag{93}$$

i.e., function $W(\rho, \theta)$, determined by formula (79) when $\rho = 1$ indeed turns into the assigned function $F(\theta)$. Assuming in (79)

$$F(\theta) = f(a\cos\theta) - P\ln\frac{2}{a}, \tag{94}$$

we will obtain function $W(\rho, \theta)$, which satisfies the differential equation (61) when $\rho < 1$ and boundary condition (62) when $\rho = 1$. Substituting (94) into (79), we will find

$$W(\rho, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{\left[f(a\cos\varphi) - P\ln\frac{2}{a}\right](1-\rho^2)}{1-2\rho\cos(\varphi-\theta)+\rho^2}\,d\varphi. \tag{95}$$

Assuming in (86) and (85) $\varepsilon = \pi$, we will find

$$a_0 = \frac{\pi}{2} \quad \text{when} \quad \varepsilon = \pi,$$
$$\int_0^{2\pi} \frac{(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2} = 2\pi. \tag{96}$$

Thus, formula (95) can be given the form

$$W(\rho, \theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{f(a\cos\varphi)(1-\rho^2)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2} - P\ln\frac{2}{a}. \tag{97}$$

Substituting (97) into (63), we will find

$$P = \frac{1}{2\pi\ln\frac{2}{a}}\int_0^{2\pi} f(a\cos\varphi)\,d\varphi. \tag{98}$$

Formulas (97) and (98) determine the unknown function $W(\rho, \theta)$, which satisfies the Laplace equation (61) when $\rho < 1$ and conditions (62)

and (63) when $\rho = 1$ and when $\rho = 0$. In order to find the unknown function $p(x)$, it remains to substitute (97) and (98) into (64). When $\rho < 1$, by fulfilling differentiation under the sign of the integral, from formula (97) we will find

$$\frac{\partial W(\rho,\theta)}{\partial\rho} = \frac{1}{\pi}\int_0^{2\pi}\frac{f(a\cos\varphi)\left[(1+\rho^2)\cos(\varphi-\theta)-2\rho\right]d\varphi}{\left[1-2\rho\cos(\varphi-\theta)+\rho^2\right]^2}. \tag{99}$$

Using identity

$$\frac{\partial}{\partial\rho}\left(\frac{\sin(\varphi-\theta)}{1-2\rho\cos(\varphi-\theta)+\rho^2}\right) = \frac{(1+\rho^2)\cos(\varphi-\theta)-2\rho}{\left[1-2\rho\cos(\varphi-\theta)+\rho^2\right]^2}.$$

and fulfilling in (99) partial integration, let us give to formula (90) the form

$$\frac{\partial W(\rho,\theta)}{\partial\rho} = \frac{a}{\pi}\int_0^{2\pi}\frac{f'(a\cos\varphi)\sin\varphi\sin(\varphi-\theta)\,d\varphi}{1-2\rho\cos(\varphi-\theta)+\rho^2} \tag{100}$$

(according to the condition function $f(x)$ is continuous and has a piecewise continuous derivative).

Formally assuming in (100) $\rho = 1$ and taking into account the identity

$$\frac{\sin(\varphi-\theta)}{1-\cos(\varphi-\theta)} = \frac{2\sin\frac{\varphi-\theta}{2}\cos\frac{\varphi-\theta}{2}}{2\sin^2\frac{\varphi-\theta}{2}} = \operatorname{ctg}\frac{\varphi-\theta}{2},$$

we will find

$$\frac{\partial W(\rho,\theta)}{\partial\rho} = \frac{a}{2\pi}\int_0^{2\pi} f'(a\cos\varphi)\sin\varphi\operatorname{ctg}\frac{\varphi-\theta}{2}\,d\varphi, \tag{101}$$

or

$$\frac{\partial W(\rho,\theta)}{\partial\rho} = -\frac{1}{2\pi}\int_0^{2\pi}\frac{df(a\cos\varphi)}{d\varphi}\operatorname{ctg}\frac{\varphi-\theta}{2}\,d\varphi. \tag{102}$$

Not dwelling now on the question about the existence of the obtained definite integral (102) and substituting (102) into (64), we will find

$$p(a\cos\vartheta)=\frac{1}{\pi a|\sin\vartheta|}\left[P-\frac{1}{2\pi}\int_0^{2\pi}\frac{df(a\cos\varphi)}{d\varphi}\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi\right]. \qquad (103)$$

Substituting P from (98) into (103) and replacing in (103) $a\cos\vartheta$ and x, let us find the unknown solution $p(x)$ of the initial equation (1).

Thus, for the desired function p we obtained two formulas (71) and (103). Let us discuss briefly their comparisons.

By differentiation with respect to ϑ, from formula (65) we will find

$$\frac{df(a\cos\vartheta)}{d\vartheta}=-\sum_{n=1}^{\infty}a_n n\sin n\vartheta. \qquad (104)$$

If function $\Phi(\vartheta)$ is presented by the Fourier series

$$\Phi(\vartheta)=a_0+\sum_{n=1}^{\infty}(a_n\cos n\vartheta+b_n\sin n\vartheta), \qquad (105)$$

then the series

$$\sum_{n=1}^{\infty}(a_n\sin n\vartheta-b_n\cos n\vartheta)$$

is called the series conjugate with series (105), and the sum of this series, if it converges, is designated by $\bar{\Phi}(\vartheta)$:

$$\bar{\Phi}(\vartheta)=\sum_{n=1}^{\infty}(a_n\sin n\vartheta-b_n\cos n\vartheta). \qquad (106)$$

Using this designation, from (104) we will find

$$\overline{\frac{df(a\cos\vartheta)}{d\vartheta}}=\sum_{n=1}^{\infty}a_n n\cos n\vartheta. \qquad (107)$$

On the basis of (107), formula (71) can be given the form

$$p(a\cos\vartheta)=\frac{1}{\pi a\,|\sin\vartheta|}\left[P-\frac{1}{2\pi}\int_0^{2\pi}\frac{df(a\cos\varphi)}{d\varphi}\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi\right], \tag{108}$$

In the thoery of trigonometric series it is proved that the sum of conjugate series $\Phi(\vartheta)$ and $\overline{\Phi(\vartheta)}$ are connected with each other by the dependence[1]

$$\overline{\Phi(\vartheta)}=-\frac{1}{2\pi}\int_0^{2\pi}\Phi(\varphi)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi. \tag{109}$$

Hence

$$\overline{\frac{df(a\cos\vartheta)}{d\vartheta}}=-\frac{1}{2\pi}\int_0^{2\pi}\frac{df(a\cos\varphi)}{d\varphi}\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi. \tag{110}$$

By substituting (110) into (108), we will obtain the formula

$$p(a\cos\vartheta)=\frac{1}{\pi a\,|\sin\vartheta|}\left[P+\overline{\frac{df(a\cos\vartheta)}{d\vartheta}}\right].$$

which coincides with formula (103). Thus, if in the general formula (108) function $\overline{\frac{df(a\cos\vartheta)}{d\vartheta}}$ is represented in the form of a trigonometric series (107), conjugate with the Fourier series (104) for function $\frac{df(a\cos\vartheta)}{d\vartheta}$, we will obtain formula (71), which determines function p in the form of an infinite series. If, however, we use the integral formula (110), we will obtain from the general formula (108) formula (103), which represents function p in the form of a definite integral. With derivation of formulas (71) and (103) we arrived at the series of (107) and the definite integral (110), formally assuming $\rho = 1$ in the first of formulas (68) and in formula (100). The question about the convergence of series (107) and the equivalent question about the existence of the definite integral (110), the integrand of which turns into infinity with $\varphi=\vartheta$, has remained open here. In

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[1]See Mikhlin, S. G., Applications of integral equations to certain problems of mechanics, mathematical physics and technology, State United Publishing Houses, 1947, p. 91.

the region of the theory of the trigonometric series, these questions have been the subject of a large quantity of investigations. Not dwelling on these extensive investigations, let us note only the following. We assume function $f'(x)$ to be piecewise-continuous. In this case series (104) will always be convergent (in particular, for points of discontinuity of function $f'(x)$ this series will give an arithmetic mean from limiting values of derivative $\frac{d f(a\cos\vartheta)}{d\vartheta}$ with an approach to the point of discontinuity on the right and on the left).

However, the convergence of series (104) does not yet entail the convergence of conjugate series (107). In particular, for points of discontinuity of function $f'(x)$, this series diverges, and, accordingly, the definite integral (110) does not have meaning. The solution $p(x)$ of the initial equation (1) infinitely increases with an approach to the point of discontinuity of function $f'(x)$. Later in § 3, we indicate certain sufficient conditions with which function $f'(x)$ should satisfy so that function $p(x)$ remains limited, and indicate the procedure of calculation of the definite integral, which determines the solution $p(x)$ of the initial equation (1).

In conclusion of this paragraph, let us indicate one more conversion of formula (103) obtained by us for function p. Assuming

$$\varphi = 2\pi - \varphi',$$

let us find

$$\frac{d f(a\cos\varphi)}{d\varphi} = -a\sin\varphi f'(a\cos\varphi) = a\sin\varphi' f'(a\cos\varphi') = -\frac{d f(a\cos\varphi')}{d\varphi'},$$

$$\operatorname{ctg}\frac{\varphi-\vartheta}{2} = \operatorname{ctg}\left(\pi - \frac{\varphi'+\vartheta}{2}\right) = -\operatorname{ctg}\frac{\varphi'+\vartheta}{2},$$

$$\int_{\pi}^{2\pi} \frac{d f(a\cos\varphi)}{d\varphi}\operatorname{ctg}\frac{\varphi-\vartheta}{2}\,d\varphi = \int_{0}^{\pi} \frac{d f(a\cos\varphi')}{d\varphi'}\operatorname{ctg}\frac{\varphi'+\vartheta}{2}\,d\varphi'. \tag{111}$$

Hence

$$\int_{0}^{2\pi} \frac{d f(a\cos\varphi)}{d\varphi}\operatorname{ctg}\frac{\varphi-\vartheta}{2}\,d\varphi = \int_{0}^{\pi} \frac{d f(a\cos\varphi)}{d\varphi}\left(\operatorname{ctg}\frac{\varphi-\vartheta}{2} + \operatorname{ctg}\frac{\varphi+\vartheta}{2}\right)d\varphi = 2\int_{0}^{\pi} \frac{d f(a\cos\varphi)}{d\varphi}\,\frac{\sin\varphi}{\cos\vartheta-\cos\varphi}\,d\varphi, \tag{112}$$

since

$$\operatorname{ctg}\frac{\varphi-\vartheta}{2} + \operatorname{ctg}\frac{\varphi+\vartheta}{2} = \frac{\cos\frac{\varphi-\vartheta}{2}\sin\frac{\varphi+\vartheta}{2} + \cos\frac{\varphi+\vartheta}{2}\sin\frac{\varphi-\vartheta}{2}}{\sin\frac{\varphi+\vartheta}{2}\sin\frac{\varphi-\vartheta}{2}} = \frac{\sin\varphi}{\frac{1}{2}(\cos\vartheta-\cos\varphi)}.$$

Substituting (112) into (103), we will find

$$p(a\cos\theta)=\frac{1}{\pi a|\sin\theta|}\left[P-\frac{1}{\pi}\int_0^{\pi}\frac{df(a\cos\varphi)}{d\varphi}\frac{\sin\varphi}{\cos\theta-\cos\varphi}d\varphi\right]. \qquad (113)$$

Assuming

$$a\cos\theta=x,\quad a\cos\varphi=t, \qquad (114)$$

we will find

$$a|\sin\theta|=\sqrt{a^2-a^2\cos^2\theta}=\sqrt{a^2-x^2},$$

$$\frac{df(a\cos\varphi)}{d\varphi}=-a\sin\varphi f'(a\cos\varphi)=-\sqrt{a^2-t^2}f'(t) \quad \text{when} \quad 0<\varphi<\pi,$$

$$p(x)=\frac{1}{\pi\sqrt{a^2-x^2}}\left[P-\frac{1}{\pi}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,dt}{t-x}\right]. \qquad (115)$$

Formula (98) can be given the form

$$P=\frac{1}{\pi\ln\frac{2}{a}}\int_0^{\pi}f(a\cos\varphi)\,d\varphi,$$

or, assuming $a\cos\varphi=t$,

$$P=\frac{1}{\pi\ln\frac{2}{a}}\int_{-a}^{a}\frac{f(t)\,dt}{\sqrt{a^2-t^2}}. \qquad (116)$$

Formulas (115) and (116) determine the solution of the initial equation (1).

§ 3. Solution of the Fundamental Equation of the Two-Dimensional Contact Problem by Means of Function of a Complex Variable

In this section we give one more derivation of formula (115), which gives the solution to equation (1). This derivative is based on elementary concepts about functions of the complex variable. Subsequently, the method of the solution of equation (1), given in this chapter, will give to us the possibility of obtaining the solution of more general equations.

If in the expression for function $F(x)$ argument x is replaced by the complex number $x + iy$, then we will obtain in general the complex number $F(x + iy)$, the real and imaginary part of which we will designate by u and v:

$$u+iv=F(x+iy). \tag{117}$$

If one were to change the real and imaginary part of the complex number $x + iy$, then in this case the real and imaginary part of the complex number $u + iv$ will be changed. In other words, u and v will be functions of two variables x and y:

$$u=u(x, y), \quad v=v(x, y). \tag{118}$$

Thus, for example, if $F(x)=x^2$, then $F(x+iy)=(x+iy)^2=x^2-y^2+2ixy$, and in this case $u=x^2-y^2$, $v=2xy$. Let us investigate properties of functions $u(x, y)$ and $v(x, y)$ definable by relation (117).

Differentiating (117), we will find

$$\left.\begin{array}{l}\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}=F'(x+iy),\\ \frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}=iF'(x+iy),\end{array}\right\} \tag{119}$$

whence

$$i\left(\frac{\partial u}{\partial x}+i\frac{\partial v}{\partial x}\right)=\frac{\partial u}{\partial y}+i\frac{\partial v}{\partial y}. \tag{120}$$

By comparing the real and imaginary parts of the right and left sides of relation (120), we will find

$$\left.\begin{array}{l}\frac{\partial v}{\partial x}=-\frac{\partial u}{\partial y},\\ \frac{\partial v}{\partial y}=\frac{\partial u}{\partial x}.\end{array}\right\} \tag{121}$$

As we already know from § 1, relation (121) indicate that function $v(x, y)$ will be the harmonic function of variables x, y, which is conjugate with the harmonic function $u(x, y)$. Thus, dependence (117) of every function $F(x)$ places in correspondence

to the pair of conjugate harmonic functions $u(x, y)$ and $v(x, y)$, which we will find by, separating the real and imaginary parts in expression $F(x + iy)$. Having designated by z the complex number $x + iy$, we will call $F(z)$ the function of the complex variable z. The real and imaginary parts of $F(z)$ will be designated by $\mathrm{Re}F(z)$ and $\mathrm{Im}F(z)$ respectively:

$$u(x, y) = \mathrm{Re}\,F(z), \quad v(x, y) = \mathrm{Im}\,F(z). \tag{122}$$

Thus, the real and imaginary parts $\mathrm{Re}F(z)$ and $\mathrm{Im}F(z)$ of the function of the complex variable $F(z)$ are conjugate harmonic functions of variables x and y.

Let us consider now the function of the complex variable z:

$$F(z) = \int_{-a}^{a} \frac{p(t)\,dt}{t-z}. \tag{123}$$

Let us find

$$F(z) = \int_{-a}^{a} \frac{p(t)\,dt}{t-x-iy} = \int_{-a}^{a} \frac{p(t)(t-x+iy)\,dt}{(t-x)^2+y^2}. \tag{124}$$

By separating the real and imaginary parts in (124), we will find

$$\left.\begin{aligned} \mathrm{Re}\,F(z) &= \int_{-a}^{a} \frac{p(t)(t-x)\,dt}{(t-x)^2+y^2}, \\ \mathrm{Im}\,F(z) &= \int_{-a}^{a} \frac{p(t)\,y\,dt}{(t-x)^2+y^2}. \end{aligned}\right\} \tag{125}$$

By comparing (125) and (5), we will find

$$\left.\begin{aligned} \mathrm{Re}\,F(z) &= \frac{\partial V(x, y)}{\partial x}, \\ \mathrm{Im}\,F(z) &= -\frac{\partial V(x, y)}{\partial y}. \end{aligned}\right\} \tag{126}$$

Thus, function $F(z)$, determined by the relation (123), is connected with the logarithmic potential of the simple layer $V(x, y)$, determined by formula (2), and relations (126), whence

$$F(z)=\frac{\partial V}{\partial x}-i\frac{\partial V}{\partial y}. \tag{127}$$

Condition (4) for function $V(x, y)$, according to (126), gives

$$\operatorname{Re} F(z)=f'(x) \text{ when } y=0, |x|<a. \tag{128}$$

The second of relations (125) directly gives

$$\operatorname{Im} F(z)=0 \text{ when } y=0, |x|>a. \tag{129}$$

Finally, from (24) and (126) it follows that

$$\left.\begin{array}{ll}\operatorname{Im} F(z)=\pi p(x) & \text{when } y=+0, |x|<a,\\ \operatorname{Im} F(z)=-\pi p(x) & \text{when } y=-0, |x|<a.\end{array}\right\} \tag{130}$$

The expansion of function $F(z)$, determined by relation (123), in series in powers of $1/z$, or, so to speak, expansion in the neighborhood of the point at infinity of plane xOy, gives

$$F(z)=-\int_{-a}^{a} p(t)\left(\frac{1}{z}+\frac{t}{z^2}+\frac{t^2}{z^3}+\ldots\right)dt=\frac{a_0}{z}+\frac{a_1}{z^2}+\frac{a_2}{z^3}+\ldots, \tag{131}$$

where $a_0, a_1, a_2, \ldots$ are real numbers, where

$$a_0=-\int_{-a}^{a} p(t)\,dt,$$

or

$$a_0=-P \tag{132}$$

according to designation (27). Thus, in the neighborhood of the point at infinity of plane xOy function $F(z)$ should have the expansion

$$F(z)=-\frac{P}{z}+\frac{a_1}{z^2}+\frac{a_2}{z^3}+\ldots \tag{133}$$

Thus, the solution of equation (1) is reduced by us to the construction of the function of the complex variable $F(z)$, which satisfies conditions (128) and (129) on axis Ox and condition (133) in the point at infinity. Having constructed function $F(z)$, we will be able to find solution $p(x)$ of equation (1) according to one of formulas (130).

Let us consider at first the simplest case in which

$$f(x) = \alpha = \text{const.} \quad \text{when} \quad |x| < a. \tag{134}$$

Designation by $F_0(z)$ the function of the complex variable z, which solve in this case the stated problem, let us find for it, according to (128) and (129), these conditions:

$$\begin{aligned} \operatorname{Re} F_0(z) &= 0 \quad \text{when} \quad y = 0, \ |x| < a, \\ \operatorname{Im} F_0(z) &= 0 \quad \text{when} \quad y = 0, \ |x| > a. \end{aligned} \tag{135}$$

Let us examine function

$$z - x_0, \tag{136}$$

where x_0 is the real constant. Let us find

$$z - x_0 = x - x_0 + iy.$$

Considering

$$x - x_0 = r\cos\varphi, \quad y = r\sin\varphi$$

(Fig. 1), let us find

$$z - x_0 = r(\cos\varphi + i\sin\varphi) = re^{i\varphi}.$$

Hence

$$\sqrt{z - x_0} = \sqrt{r}e^{i\frac{\varphi}{2}}. \tag{137}$$

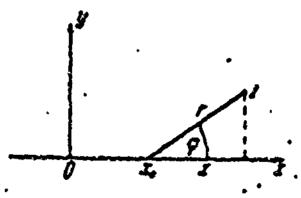

Fig. 1.

For definiteness we will consider that

$$\varphi = 0 \text{ when } y = 0,\ x > x_0 \tag{138}$$

and further (Fig. 1):

$$\left.\begin{aligned} \varphi &= \pi \quad \text{when } y = +0,\ x < x_0, \\ \varphi &= -\pi \quad \text{when } y = -0,\ x < x_0. \end{aligned}\right\} \tag{139}$$

At the same time

$$r = |x - x_0| \text{ when } y = 0. \tag{140}$$

From relations (137), (139) and (140) it follows that

$$\left.\begin{aligned} \sqrt{z - x_0} &= \sqrt{x - x_0} \quad \text{when } y = 0,\ x > x_0, \\ \sqrt{z - x_0} &= i\sqrt{x_0 - x} \quad \text{when } y = +0,\ x < x_0, \\ \sqrt{z - x_0} &= -i\sqrt{x_0 - x} \quad \text{when } y = -0,\ x < x_0, \end{aligned}\right\} \tag{141}$$

since

$$e^{i\frac{\pi}{2}} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i, \quad e^{-i\frac{\pi}{2}} = \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i.$$

Assuming in (141) $x_0 = a$ and $x_0 = -a$, we will find

$$\begin{aligned} &\sqrt{z - a} = \sqrt{x - a},\ x > a, \\ &\sqrt{z + a} = \sqrt{x + a},\ x > -a,\ y = 0, \\ &\sqrt{z - a} = i\sqrt{a - x},\ x < a, \\ &\sqrt{z + a} = i\sqrt{-a - x},\ x < -a,\ y = +0, \\ &\sqrt{z - a} = -i\sqrt{a - x},\ x < a, \\ &\sqrt{z + a} = -i\sqrt{-a - x},\ x < -a,\ y = -0, \end{aligned}$$

whence

$$\left.\begin{aligned} \sqrt{z^2 - a^2} &= \sqrt{x^2 - a^2}, & x > a,\ y = 0, \\ \sqrt{z^2 - a^2} &= i\sqrt{a^2 - x^2}, & -a < x < a,\ y = +0, \\ \sqrt{z^2 - a^2} &= -i\sqrt{a^2 - x^2}, & -a < x < a,\ y = -0, \\ \sqrt{z^2 - a^2} &= -\sqrt{x^2 - a^2}, & x < -a,\ y = 0. \end{aligned}\right\} \tag{142}$$

Let us now show that function

$$F_0(z) = -\frac{P}{\sqrt{z^2-a^2}}. \tag{143}$$

satisfies all the set conditions.

Actually,

$$F_0(z) = -\frac{P}{z}\left(1-\frac{a^2}{z^2}\right)^{-\frac{1}{2}} = -\frac{P}{z}\left(1+\frac{a^2}{2z^2}+\frac{3a^4}{8z^4}+\ldots\right) = -\frac{P}{z}-\frac{Pa^2}{2z^3}-\frac{3Pa^4}{8z^5}-\ldots, \tag{144}$$

i.e., condition (133) is fulfilled. Substituting (142) into (143), we will find

$$\left.\begin{aligned} &F_0(z) = -\frac{P}{\sqrt{x^2-a^2}}, \quad x > a, \\ &F_0(z) = \frac{P}{\sqrt{x^2-a^2}}, \quad x < -a, \quad y = 0, \end{aligned}\right\} \tag{145}$$

$$\left.\begin{aligned} &F_0(z) = \frac{iP}{\sqrt{a^2-x^2}}, \quad y = +0, \\ &F_0(z) = -\frac{iP}{\sqrt{a^2-x^2}}, \quad y = -0, \; |x| < a. \end{aligned}\right\} \tag{146}$$

Relations (145) and (146) show that condition (135) is fulfilled.

By comparing (130) and (146), we will find

$$p(x) = \frac{P}{\pi\sqrt{a^2-x^2}}, \tag{147}$$

which coincides with the earlier found solution (74).

Passing to the general case, we will look for function $F(z)$, which satisfies conditions (128), (129) and (133), in the form

$$F(z) = F_0(z)\cdot\Phi(z), \tag{148}$$

bringing the determination of function $F(z)$ to the finding to the finding of function $\Phi(z)$. As one can see from (146), when $|x|<a$, $y=\pm 0$ function $F_0(z)$ is pure imaginary. Consequently.

$$\operatorname{Re} F(z)=iF_0(z)\operatorname{Im}\Phi(z) \quad \text{when} \quad |x|<a,\ y=\pm 0. \tag{149}$$

Substituting into (149), (128) and (146), we will find

$$f'(x)=-\frac{P}{\sqrt{a^2-x^2}}\operatorname{Im}\Phi(z), \quad y=+0,\ |x|<a,$$
$$f'(x)=\frac{P}{\sqrt{a^2-x^2}}\operatorname{Im}\Phi(z), \quad y=-0,\ |x|<a,$$

whence

$$\left.\begin{aligned}\operatorname{Im}\Phi(z)&=-\frac{1}{P}\sqrt{a^2-x^2}\,f'(x), \quad |x|<a,\ y=+0,\\ \operatorname{Im}\Phi(z)&=\frac{1}{P}\sqrt{a^2-x^2}\,f'(x), \quad |x|<a,\ y=-0.\end{aligned}\right\} \tag{150}$$

As can be seen from (145), when $|x|>a$, $y=0$ function $F_0(z)$ is real. Consequently,

$$\operatorname{Im} F(z)=F_0(z)\operatorname{Im}\Phi(z) \quad \text{when} \quad |x|>a,\ y=0. \tag{151}$$

Substituting (129) and (145) into (151), we will find

$$\operatorname{Im}\Phi(z)=0 \quad \text{when} \quad |x|>a,\ y=0. \tag{152}$$

We will look for $\Phi(z)$, in the form

$$\Phi(z)=\int_{-a}^{a}\frac{g(t)\,dt}{t-z}+c, \tag{153}$$

where c – real constant. Using formulas (123), (129) and (130), we will find

$$\left.\begin{array}{lll}\operatorname{Im}\Phi(z)=\pi q(x) & \text{when} & |x|<a,\ y=+0,\\ \operatorname{Im}\Phi(z)=-\pi q(x) & \text{when} & |x|<a,\ y=-0,\\ \operatorname{Im}\Phi(z)=0 & \text{when} & |x|>a,\ y=0.\end{array}\right\} \tag{154}$$

Comparing (150) and (152) with (154), we will arrive at the conclusion that function $\Phi(z)$ determined by the relation (153), will satisfy the set conditions (150) and (152) if one were to assume

$$q(x)=-\frac{1}{\pi p}\sqrt{a^2-x^2}\,f'(x). \tag{155}$$

Substituting (155) into (153), we will find

$$\Phi(z)=-\frac{1}{\pi p}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,dt}{t-z}+c. \tag{156}$$

Substituting (143) and (156) into (148), we will find

$$F(z)=\frac{1}{\pi\sqrt{z^2-a^2}}\left[\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,dt}{t-z}-\pi pc\right]. \tag{157}$$

Thus, function $F(z)$, determined by relation (157), satisfies conditions (129) and (130). Expanding this function in series in powers of $1/z$, we will find

$$F(z)=\frac{1}{\pi z}\left(1+\frac{a^2}{2z^2}+\ldots\right)\left(\frac{b_0}{z}+\frac{b_1}{z^2}+\ldots-\pi pc\right)=$$
$$=-\frac{pc}{z}+\frac{b_0}{\pi z^2}+\ldots, \tag{158}$$

where $b_0,\ b_1,\ b_2,\ldots$ – certain real coefficients. Comparing (133) and (158), we will arrive at conclusion that condition (133) will be fulfilled if one were to assume

$$c=1. \tag{159}$$

Substituting (159) in (157), we will find

$$F(z)=\frac{1}{\pi\sqrt{z^2-a^2}}\left[\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,dt}{t-z}-\pi p\right]. \tag{160}$$

In order to find the unknown $p(x)$, it remains to substitute (160) into (130). According to (142) function $\sqrt{z^2-a^2}$ when $|x|<a,\ y=\pm 0$ is pure imaginary. Consequently,

$$\operatorname{Im} F(z) = \frac{1}{\pi i\sqrt{z^2-a^2}}\left[\operatorname{Re}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,dt}{t-z}-\pi P\right] \tag{161}$$
$$\text{when } |x|<a,\ y=\pm 0.$$

Taking into account the relation

$$\operatorname{Re}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,dt}{t-z}=\operatorname{Re}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,(t-x+iy)\,dt}{(t-x)^2+y^2}=$$
$$=\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,(t-x)\,dt}{(t-x)^2+y^2},$$

and relation (142), we will find

$$\left.\begin{aligned}
&\operatorname{Im} F(z)=\\
&\quad=-\frac{1}{\pi\sqrt{a^2-x^2}}\left[\lim_{y\to+0}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,(t-x)\,dt}{(t-x)^2+y^2}-\pi P\right]\\
&\qquad\text{при } |x|<a,\ y=+0,\\
&\operatorname{Im} F(z)=\\
&\quad=\frac{1}{\pi\sqrt{a^2-x^2}}\left[\lim_{y\to-0}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}\,(t-x)\,dt}{(t-x)^2+y^2}-\pi P\right].\\
&\qquad\text{при } |x|<a,\ y=-0.
\end{aligned}\right\} \tag{162}$$

Formally passing to the limit in formulas (162), i.e., considering in definite integrals entering into this formula $y = 0$ and comparing (130) and (162), we will find

$$p(x)=\frac{1}{\pi\sqrt{a^2-x^2}}\left[P-\frac{1}{\pi}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt\right], \tag{163}$$

which completely coincides with formula (115) obtained in § 2.

Let us dwell at this time more specifically on the last passage to the limit, which leads us to formula (163).

Let us divide the definite integral

$$J(x,\ y)=\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}(t-x)\,dt}{(t-x)^2+y^2}\tag{164}$$

into four components:

$$J(x,\ y)=J_1(x,\ y)+J_2(x,\ y)+J_3(x,\ y)+J_4(x,\ y),\tag{164'}$$

where

$$J_1(x,\ y)=\int_{-a}^{x-\varepsilon}\frac{f'(t)\sqrt{a^2-t^2}(t-x)\,dt}{(t-x)^2+y^2},\tag{165}$$

$$J_2(x,\ y)=\int_{x+\varepsilon}^{a}\frac{f'(t)\sqrt{a^2-t^2}(t-x)\,dt}{(t-x)^2+y^2},\tag{166}$$

$$J_3(x,\ y)=\int_{x-\varepsilon}^{x+\varepsilon}\frac{[f'(t)\sqrt{a^2-t^2}-f'(x)\sqrt{a^2-x^2}](t-x)\,dt}{(t-x)^2+y^2},\tag{167}$$

$$J_4(x,\ y)=f'(x)\sqrt{a^2-x^2}\int_{x-\varepsilon}^{x+\varepsilon}\frac{(t-x)\,dt}{(t-x)^2+y^2}.\tag{168}$$

In definite integrals (165) when $y = 0$, the integrands remain limited in the whole region of integration. Thus,

$$J_1(x,\ 0)=\int_{-a}^{x-\varepsilon}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt,\quad J_2(x,\ 0)=\int_{x+\varepsilon}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt.\tag{169}$$

Further

$$\int_{x-\varepsilon}^{x+\varepsilon}\frac{(t-x)\,dt}{(t-x)^2+y^2}=\frac{1}{2}\ln\left[(t-x)^2+y^2\right]_{t=x-\varepsilon}^{t=x+\varepsilon}=$$
$$=\frac{1}{2}\left[\ln(\varepsilon^2+y^2)-\ln(\varepsilon^2+y^2)\right]=0,$$

whence

$$J_4(x,\ y)=0,\qquad y>0 \text{ и } y<0.\tag{170}$$

On the basis (169) and (170), from formula (164') we will find

$$J(x,\ \pm 0)-\int_{-a}^{x-\varepsilon}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt-\int_{x+\varepsilon}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt=$$
$$=J_3(x,\ \pm 0);\tag{171}$$

It remains to investigate the limiting values $J_3(x, +0)$ and $J_3(x, -0)$ of the definite integral $J_3(x, y)$. It is said that function $\varphi(t)$ at point $t = x$ satisfies conditions of Lipschitz, if in a sufficiently small interval $(x-\varepsilon, x+\varepsilon)$ it is possible to select such two positive constants M and α $(0<\alpha\leqslant 1)$, so that inequality is fulfilled

$$|\varphi(t)-\varphi(x)|<M|t-x|^{\alpha} \text{ when } x-\varepsilon\leqslant t\leqslant x+\varepsilon. \tag{172}$$

If the two functions $\varphi_1(t)$ and $\varphi_2(t)$ satisfy the condition of Lipschitz at point $t = x$, *i.e.*,

$$\begin{gathered}|\varphi_1(t)-\varphi_1(x)|<M_1|t-x|^{\alpha_1}, \quad |\varphi_2(t)-\varphi_2(x)|<M_2|(t-x)|^{\alpha_2}, \\ 0<\alpha_1\leqslant 1, \quad 0<\alpha_2\leqslant 1, \quad x-\varepsilon\leqslant t\leqslant x+\varepsilon,\end{gathered} \tag{173}$$

then the product of them

$$\varphi(t)=\varphi_1(t)\varphi_2(t) \tag{174}$$

will also satisfy the condition of Lipschitz. Actually,

$$\begin{aligned}\varphi(t)-\varphi(x)&=\varphi_1(t)\varphi_2(t)-\varphi_1(x)\varphi_2(x)=\\ &=[\varphi_1(t)-\varphi_1(x)]\varphi_2(x)+[\varphi_2(t)-\varphi_2(x)]\varphi_1(t),\end{aligned}$$

whence on the basis of (173)

$$\begin{gathered}|\varphi(t)-\varphi(x)|\leqslant M_1|t-x|^{\alpha_1}|\varphi_2(x)|+M_2|t-x|^{\alpha_2}m_1\leqslant M|t-x|^{\alpha}, \\ x-\varepsilon\leqslant t\leqslant x+\varepsilon,\end{gathered} \tag{175}$$

where m_1 – maximum absolute value of function $\varphi_1(t)$ in the interval $(x-\varepsilon, x+\varepsilon)$, $M=M_1|\varphi_2(x)|+M_2m_1$, α – is the smaller of numbers α_1 and α_2 (in (175) we assume that ε in any case is less than unity).

Thus, let us assume that function $f'(t)$ satisfies at point $t=x$ $(-a<x<a)$ the condition of Lipschitz. Then function $f'(t)\sqrt{a^2-t^2}$ will satisfy the condition of Lipschitz at this point, since when

$-a < x < a\ \sqrt{a^2-t^2}$ satisfies the condition of Lipschitz with α equal to unity:

$$\left|\sqrt{a^2-t^2}-\sqrt{a^2-x^2}\right| \leqslant m|t-x|, \quad x-\varepsilon \leqslant t \leqslant x+\varepsilon,$$

were m – the largest absolute value of the derivative with respect to t of $\sqrt{a^2-t^2}$ in the interval $(x-\varepsilon, x+\varepsilon)$. Thus, the inequality will take place

$$\left|f'(t)\sqrt{a^2-t^2}-f'(x)\sqrt{a^2-x^2}\right| \leqslant M|t-x|^{\alpha}, \quad 0 < \alpha \leqslant 1,$$
$$x-\varepsilon \leqslant t \leqslant x+\varepsilon. \tag{176}$$

On the basis (176) from (167) we find:

$$|J_2(x, y)| \leqslant M\int_{x-\varepsilon}^{x+\varepsilon}\frac{|t-x|^{1+\alpha}dt}{(t-x)^2+y^2}, \quad 0 < \alpha \leqslant 1. \tag{177}$$

When $y = 0$ the integrand in (177) turns into $|t-x|^{\alpha-1}$ and remains integrable, since $0 < \alpha \leqslant 1$. Thus,

$$|J_2(x, \pm 0)| \leqslant M\int_{x-\varepsilon}^{x+\varepsilon}|t-x|^{\alpha-1}dt =$$
$$= 2M\int_{x}^{x+\varepsilon}(t-x)^{\alpha-1}dt = \frac{2M}{\alpha}(t-x)^{\alpha}\Big|_{t=x}^{t=x+\varepsilon} = \frac{2M}{\alpha}\varepsilon^{\alpha}. \tag{178}$$

On the basis (178) from (171) we find

$$\left|J(x, \pm 0)-\int_{-a}^{x-\varepsilon}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}dt-\int_{x+\varepsilon}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}dt\right| \leqslant \frac{2M}{\alpha}\varepsilon^{\alpha}. \tag{179}$$

From (179) it is clear that $J(x, y)$ remains limited when y approaches zero, where

$$J(x, \pm 0) = \lim_{\varepsilon\to 0}\left(\int_{-a}^{x-\varepsilon}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}dt+\int_{x+\varepsilon}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}dt\right). \tag{180}$$

The expression standing in the right side of equality (180) is called the main value of the integral

$$\int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt, \tag{181}$$

and we use for it the standard designation of the definite integral, stipulating the fact that the written integral should be understood as its principal value, i.e., the limit of the sum of two parts of this integral standing in the right side of equality (180).

Thus, on the basis of (180) from (164) we find

$$\lim_{y\to\pm 0} \int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}\,(t-x)\,dt}{(t-x)^2+y^2} = \int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt, \tag{182}$$

if function $f'(t)$ satisfies the condition of Lipschitz at point $t=x\ (-a<x<a)$, where the definite integral standing in the right side of relation (182) should be understood as its principal value, i.e., the limit shown in the right side of formula (180).

Substituting (182) into (162), we will find

$$\left.\begin{aligned} &\operatorname{Im} F(z) = -\frac{1}{\pi\sqrt{a^2-x^2}}\left(\int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x} - \pi P\right) \\ &\qquad \text{when } |x|<a,\ y=+0, \\ &\operatorname{Im} F(z) = \frac{1}{\pi\sqrt{a^2-x^2}}\left(\int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x} - \pi P\right) \\ &\qquad \text{when } |x|<a,\ y=-0, \end{aligned}\right\} \tag{183}$$

where, according to (130),

$$p(x) = \frac{1}{\pi\sqrt{a^2-x^2}}\left[P - \frac{1}{\pi}\int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt\right]. \tag{184}$$

Thus, for the case when at point $t=x\ (-a<x<a)$ function $f'(t)$ satisfies the condition of Lipschitz, we justified the derivation of formula (163), obtained by us earlier as a result of the formal

passage to the limit, and proved that function $p(x)$ defined by formula (163) will be in this case limited, and the definite integral entering into formula (163) can be understood as its principal value, and it must be calculated, as the limit of the sum

$$\int_{-a}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt = \lim_{\varepsilon\to 0}\left(\int_{-a}^{x-\varepsilon} \frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt + \int_{x+\varepsilon}^{a} \frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt\right). \tag{185}$$

Let us examine as an example the case when

$$f(x) = \alpha - A|x|^{k+1} \quad (0 \leqslant k \leqslant 1), \tag{186}$$

i.e.,

$$f(x) = \alpha - Ax^{k+1} \text{ when } x > 0 \text{ и } f(x) = \alpha - A(-x)^{k+1} \text{ when } x < 0. \tag{187}$$

Differentiating (187), we will find

$$f'(x) = -A(k+1)x^k \text{ when } x > 0,\ f'(x) = A(k+1)(-x)^k \text{ when } x < 0. \tag{188}$$

Substituting (188) into (184), we will find

$$p(x) = \frac{1}{\pi\sqrt{a^2-x^2}}\left(P + A\frac{k+1}{\pi}\int_{-a}^{a} \frac{\pm|t|^k\sqrt{a^2-t^2}}{t-x}\,dt\right), \tag{189}$$

where under the integral sign in (189) the plus sign can be taken when $t > 0$ and minus sign when $t < 0$, and this integral should be understood as its principal value. As can be seen from (188), when $x \neq 0$ function $f'(x)$ has a continuous derivative and satisfies the condition of Lipschitz with the index $\alpha = 1$. When $x = 0$ function $f'(x)$ satisifes the condition of Lipschitz with index $\alpha = k$ if $0 < k \leqslant 1$, and does not satisfy the condition of Lipschitz if $k = 0$. In particular, when $k = 0$ function $f'(x)$ at point $x = 0$ undergoes discontinuity:

$$f'(x) = -A \text{ when } x > 0, \quad f'(x) = A \text{ when } x < 0.$$

Thus, if $0 < k < 1$, then function $p(x)$, determined by formula (189), will be limited everywhere inside the interval $(-a, a)$. When $k = 0$ we can only affirm that function $p(x)$ will be limited when $-a < x < 0$ and $0 < x < a$.

Let us conduct computation to the end for the maximum case $k = 0$. In this case formula (189) takes the form

$$p(x) = \frac{1}{\pi\sqrt{a^2 - x^2}}\left(P + \frac{A}{\pi}\int_{-a}^{a} \frac{\pm\sqrt{a^2 - t^2}}{t - x}\,dt\right). \tag{190}$$

Let us find

$$\begin{gathered}\int_{-a}^{a} \frac{\pm\sqrt{a^2 - t^2}}{t - x}\,dt = \int_{-a}^{a} \frac{a^2 - x^2 + x^2 - t^2}{\pm\sqrt{a^2 - t^2}\,(t - x)}\,dt = \\ = \frac{a^2 - x^2}{x}\int_{-a}^{a} \frac{dt}{\pm\sqrt{a^2 - t^2}\left(\frac{t}{x} - 1\right)} - \int_{-a}^{a} \frac{t + x}{\pm\sqrt{a^2 - t^2}}\,dt.\end{gathered} \tag{191}$$

Assuming further

$$t = \frac{2a\tau}{1 + \tau^2}, \quad x = \frac{2a\xi}{1 + \xi^2}\left(\tau = \frac{a - \sqrt{a^2 - t^2}}{t}, \quad \xi = \frac{a - \sqrt{a^2 - x^2}}{x}\right), \tag{192}$$

we will find

$$\begin{gathered}dt = \frac{2a(1 - \tau^2)\,d\tau}{(1 + \tau^2)^2}, \\ \sqrt{a^2 - t^2} = \sqrt{a^2 - \frac{4a^2\tau^2}{(1 + \tau^2)^2}} = \sqrt{\frac{a^2(1 - \tau^2)^2}{(1 + \tau^2)^2}} = \frac{a(1 - \tau^2)}{1 + \tau^2}, \\ \frac{t}{x} - 1 = \frac{\tau(1 + \xi^2)}{\xi(1 + \tau^2)} - 1 = -\frac{(\tau - \xi)(\xi\tau - 1)}{\xi(1 + \tau^2)} = -\frac{(\tau - \xi)\left(\tau - \frac{1}{\xi}\right)}{1 + \tau^2},\end{gathered}$$

whence

$$\begin{aligned}\frac{dt}{\sqrt{a^2 - t^2}\left(\frac{t}{x} - 1\right)} &= -2\frac{d\tau}{(\tau - \xi)\left(\tau - \frac{1}{\xi}\right)} = \\ &= \frac{2\xi}{1 - \xi^2}\left(\frac{1}{\tau - \xi} + \frac{1}{\frac{1}{\xi} - \tau}\right)d\tau = \\ &= \frac{2\xi}{1 - \xi^2}\,d\ln\frac{\tau - \xi}{\frac{1}{\xi} - \tau} = \frac{2\xi}{1 - \xi^2}\,d\ln\frac{\xi - \tau}{\frac{1}{\xi} - \tau}.\end{aligned} \tag{193}$$

y finding the principal value of the definite integral

$$\int_{-a}^{a} \frac{dt}{\pm\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)} \quad \text{when } 0<x<a,$$

e will obtain

$$\int_{-a}^{a} \frac{dt}{\pm\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)} = -\int_{-a}^{0} \frac{dt}{\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)} + \lim_{\varepsilon\to 0}\left[\int_{0}^{x-\varepsilon} \frac{dt}{\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)} + \int_{x+\varepsilon}^{a} \frac{dt}{\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)}\right]. \tag{194}$$

aking into account (193) and (192), we find

$$\begin{aligned}
\int_{-a}^{a} \frac{dt}{\pm\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)} &= \\
&= \frac{2\xi}{1-\xi^2}\left[\ln \frac{\xi-\tau}{\frac{1}{\xi}-\tau}\Bigg|_{\tau=0}^{\tau=-1} + \lim_{\varepsilon\to 0}\left(\ln \frac{\xi-\tau}{\frac{1}{\xi}-\tau}\Bigg|_{\tau=0}^{\tau=\tau_1} + \ln \frac{\tau-\xi}{\frac{1}{\xi}-\tau}\Bigg|_{\tau=\tau_2}^{\tau=1}\right)\right] = \\
&= \frac{2\xi}{1-\xi^2} \lim_{\varepsilon\to 0} \ln \frac{(1+\xi)\frac{1}{\xi}(\xi-\tau_1)\frac{1}{\xi}(1-\xi)\left(\frac{1}{\xi}-\tau_2\right)}{\left(1+\frac{1}{\xi}\right)\xi\left(\frac{1}{\xi}-\tau_1\right)\xi\left(\frac{1}{\xi}-1\right)(\tau_2-\xi)} = \\
&= \frac{2\xi}{1-\xi^2}\left[\ln \frac{1}{\xi^2} + \lim_{\varepsilon\to 0} \ln \frac{(\xi-\tau_1)\left(\frac{1}{\xi}-\tau_2\right)}{\left(\frac{1}{\xi}-\tau_1\right)(\tau_2-\xi)}\right],
\end{aligned} \tag{195}$$

here

$$\tau_1 = \frac{a-\sqrt{a^2-(x-\varepsilon)^2}}{x-\varepsilon}, \quad \tau_2 = \frac{a-\sqrt{a^2-(x+\varepsilon)^2}}{x+\varepsilon}. \tag{196}$$

ccording to (192) and (196) we find

$$\lim_{\varepsilon\to 0}\ln\frac{(\xi-\tau_1)\left(\frac{1}{\xi}-\tau_2\right)}{\left(\frac{1}{\xi}-\tau_1\right)(\tau_2-\xi)}=$$

$$=\lim_{\varepsilon\to 0}\ln\frac{\left[\frac{a-\sqrt{a^2-x^2}}{x}-\frac{a-\sqrt{a^2-(x-\varepsilon)^2}}{x-\varepsilon}\right]}{\left[\frac{x}{a-\sqrt{a^2-x^2}}-\frac{a-\sqrt{a^2-(x-\varepsilon)^2}}{x-\varepsilon}\right]}\times$$

$$\times\frac{\left[\frac{x}{a-\sqrt{a^2-x^2}}-\frac{a-\sqrt{a^2-(x+\varepsilon)^2}}{x+\varepsilon}\right]}{\left[\frac{a-\sqrt{a^2-(x+\varepsilon)^2}}{x+\varepsilon}-\frac{a-\sqrt{a^2-x^2}}{x}\right]}=$$

$$=\ln\lim_{\varepsilon\to 0}\frac{-a\varepsilon-(x-\varepsilon)\sqrt{a^2-x^2}+x\sqrt{a^2-(x-\varepsilon)^2}}{-x\sqrt{a^2-(x+\varepsilon)^2}-a\varepsilon+(x+\varepsilon)\sqrt{a^2-x^2}}=$$

$$=\ln\frac{-a+\sqrt{a^2-x^2}+\frac{x^2}{\sqrt{a^2-x^2}}}{-a+\sqrt{a^2-x^2}+\frac{x^2}{\sqrt{a^2-x^2}}}=0. \qquad (197)$$

Substituting (197) into (195) and replacing after that ξ by x according to (192), let us find

$$\int_{-a}^{a}\frac{dt}{\pm\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)}=-\frac{4\xi}{1-\xi^2}\ln\xi=$$

$$=4\frac{a-\sqrt{a^2-x^2}}{x\left[1-\frac{(a-\sqrt{a^2-x^2})^2}{x^2}\right]}\ln\frac{a-\sqrt{a^2-x^2}}{x}=$$

$$=-\frac{2x}{\sqrt{a^2-x^2}}\ln\frac{a-\sqrt{a^2-x^2}}{x},\quad 0<x<a. \qquad (198)$$

By conducting analogous calculations for the case $-a<x<0$, find

$$\int_{-a}^{a}\frac{dt}{\pm\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)}=-\frac{2x}{\sqrt{a^2-x^2}}\ln\frac{a-\sqrt{a^2-x^2}}{-x};\quad -a<x<0. \qquad (199)$$

Formulas (198) and (199) can be united into one:

$$\int_{-a}^{a}\frac{dt}{\pm\sqrt{a^2-t^2}\left(\frac{t}{x}-1\right)}=-\frac{2x}{\sqrt{a^2-x^2}}\ln\frac{a-\sqrt{a^2-x^2}}{|x|},\quad 0<|x|<a. \qquad (200)$$

Further,

$$\int_{-a}^{a} \frac{(t+x)\,dt}{\pm\sqrt{a^2-t^2}} = \int_{-a}^{a} \frac{t\,dt}{\pm\sqrt{a^2-t^2}} + x\int_{-a}^{a} \frac{dt}{\pm\sqrt{a^2-t^2}} =$$

$$= 2\int_{0}^{a} \frac{t\,dt}{\sqrt{a^2-t^2}} = 2\sqrt{a^2-t^2}\Big|_{a}^{0} = 2a. \tag{201}$$

Substituting (200) and (201) into (191) and (190), we will find

$$p(x) = \frac{1}{\pi\sqrt{a^2-x^2}}\left(P - \frac{2aA}{\pi} - \frac{2A}{\pi}\sqrt{a^2-x^2}\ln\frac{a-\sqrt{a^2-x^2}}{|x|}\right),$$
$$0<|x|<a. \tag{202}$$

As can be seen from (202), the found solution of equation (1) approaches infinity when x approaches zero. When $|x|\to a$, the found function $p(x)$ also approaches infinity, with the exception of the case when

$$P = \frac{2A}{\pi}a. \tag{203}$$

In this special case

$$p(x) = -\frac{2A}{\pi^2}\ln\frac{a-\sqrt{a^2-x^2}}{|x|}, \quad 0<|x|<a, \tag{204}$$

and turns into zero when $|x|=a$.

Substituting (187) into (116), we will find

$$P = \frac{1}{\pi\ln\frac{2}{a}}\left(a\int_{-a}^{a}\frac{dt}{\sqrt{a^2-t^2}} - A\int_{-a}^{a}\frac{\pm t\,dt}{\sqrt{a^2-t^2}}\right). \tag{205}$$

Further,

$$\int_{-a}^{a}\frac{dt}{\sqrt{a^2-t^2}} = \arcsin\frac{t}{a}\Big|_{-a}^{a} = \pi,$$

$$\int_{-a}^{a}\frac{\pm t\,dt}{\sqrt{a^2-t^2}} = 2\int_{0}^{a}\frac{t\,dt}{\sqrt{a^2-t^2}} = 2\sqrt{a^2-t^2}\Big|_{a}^{0} = 2a. \tag{206}$$

Substituting (206) into (205), we will find

$$p = \frac{1}{\ln\frac{2}{a}}\left(\alpha - \frac{2Aa}{\pi}\right). \qquad (207)$$

By comparing (203) and (207), we will find that conditi·n (203) under which the initial equation (1) has solution (204˙ will be fulfilled in the case when equality will take place

$$\frac{2Aa}{\pi}\left(1 + \ln\frac{2}{a}\right) = \alpha. \qquad (208)$$

Formulas (202) and (207) solve the examined problem in general when arbitrary value of constants a, α and A; if, however, condition (208) is fulfilled, formulas (202) and (207) pass, respectively, into formulas (204) and (203).

§ 4. Case of Several Sections of Integration in the Basic Equation of the Contact Problem

Let us now examine the solution of an equation more general than equation (1):

$$\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-x|}\,dt = f(x), \quad a_m < x < b_m \quad (m = 1, 2, \ldots, n), \qquad (209)$$

where $f(x)$ – function assigned in n intervals of argument x, $a_m < x < b_m$ $(m = 1, 2, \ldots, n)$, and $p(x)$ – unknown function, subject to the determination in these n intervals of the argument. In the special case when $n = 1$, $a_1 = -a$, $b_1 = a$, equation (209) turns into equation (1).

Let us consider the function of a complex variable

$$F(z) = \sum_{m=1}^{n} F_m(z), \qquad (210)$$

here

$$F_m(z) = \int_{a_m}^{b_m} \frac{p(t)\,dt}{t-z} \quad (m = 1, 2, \ldots, n). \tag{211}$$

ccording to (127)

$$F_m(z) = \frac{\partial V_m}{\partial x} - i\,\frac{\partial V_m}{\partial y}, \tag{212}$$

here

$$V_m(x, y) = \int_{a_m}^{b_m} p(t) \ln \frac{1}{\sqrt{(t-x)^2 + y^2}}\,dt \quad (m = 1, 2, \ldots, n). \tag{213}$$

quation (209) is equivalent to condition

$$\sum_{m=1}^{n} V_m(x, 0) = f(x), \; a_m < x < b_m \quad (m = 1, 2, \ldots, n), \tag{214}$$

mposed on function $V_1(x, y), V_2(x, y), \ldots, V_n(x, y)$, or to condition

$$\operatorname{Re} F(z) = \sum_{m=1}^{n} \operatorname{Re} F_m(z) = \sum_{m=1}^{n} \frac{\partial V_m}{\partial x}(x, 0) = f'(x) \tag{215}$$

hen

$$y = 0, \qquad a_m < x < b_m \; (m = 1, 2, \ldots, n),$$

mposed on function $F(z)$.

Relation (219) and (130) will correspond to relation

$$\left.\begin{aligned} &\operatorname{Im} F_m(z) = 0 && \text{when } y = 0,\ x < a_m \text{ и } x > b_m, \\ &\operatorname{Im} F_m(z) = \pm \pi p(x) && \text{when } y = \pm 0,\ a_m < x < b_m \end{aligned}\right\} (m = 1, 2, \ldots, n),$$

whence

$$\operatorname{Im} F(z) = \sum_{m=1}^{n} \operatorname{Im} F_m(z) = 0 \quad \ldots \; y = 0,$$

$$x < a_1, \quad b_m < x < a_{m+1} \quad (m = 1, 2, \ldots, n-1), \quad b_n < x, \tag{216}$$

$$\operatorname{Im} F(z) = \pm \pi p(x) \; \ldots \; y = \pm 0, \; a_m < x < b_m \; (m = 1, 2, \ldots, n).$$

since when $y=\pm 0$, $a_m<x<b_m$, $\operatorname{Im} F_k(z)=0$ when $k\neq m$ and $\operatorname{Im} F_k(z)=\pm\pi p(x)$ when $k=m$.

According to (133) the expansion of function $F_m(z)$ in the surrounding point at infinity will have the form

$$F_m(z)=-\frac{P_m}{z}+\frac{a_{1m}}{z^2}+\ldots, \quad \text{где} \quad P_m=\int_{a_m}^{b_m} p(t)\,dt,$$

whence

$$F(z)=\sum_{m=1}^{n} F_m(z)=-\frac{P}{z}+\frac{a_1}{z^2}+\ldots, \tag{218}$$

where

$$P=\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt.$$

Thus, having constructed function $F(z)$, which satisfies conditions (215) and (216) on the axis Ox and condition (218) in the neighborhood of the point at infinity, we will find the unknown function $p(x)$ by formula (217). For simplicity we will limit ourselves to the consideration of function $F(z)$ in the upper half-plane $y > 0$, and we will determine function $p(x)$ according to (217) by the formula

$$p(x)=\frac{1}{\pi}\operatorname{Im} F(z) \quad \text{when} \quad y=+0;\ a_m<x<b_m \quad (m=1, 2, \ldots, n). \tag{219}$$

Just as in § 3, we will examine first the special case of the problem in which $f(x)$ = const, and accordingly

$$f'(x)=0 \quad \text{when} \quad a_m<x<b_m \quad (m=1, 2, \ldots, n). \tag{220}$$

Let us show that in this case function

$$F_0(z)=\frac{P_{n-1}(z)}{\sqrt{(z-a_1)(z-a_2)\ldots(z-a_n)(z-b_1)(z-b_2)\ldots(z-b_n)}}, \tag{221}$$

where $P_{n-1}(z)$ — polynomial of power $n - 1$ in z with real coefficients

$$P_{n-1}(z)=c_0+c_1z+c_2z^2+\ldots+c_{n-1}z^{n-1}, \tag{222}$$

satisfies conditions (215) and (216), i.e., let us show

$$\left.\begin{array}{c}\operatorname{Re}F_0(z)=0 \quad \text{when } a_m<x<b_m\ (m=1,2,\ldots,n),\ y=+0,\\ \operatorname{Im}F_0(z)=0 \quad \text{when } x<a_1,\\ b_m<x<a_{m+1}\ (m=1,2,\ldots,n-1),\ b_n<x,\ y=+0.\end{array}\right\} \tag{223}$$

According to (141)

$$\left.\begin{array}{ll}\dfrac{1}{\sqrt{z-a_m}}=\dfrac{1}{\sqrt{x-a_m}} & \text{when } x>a_m,\\ \dfrac{1}{\sqrt{z-a_m}}=-\dfrac{i}{\sqrt{a_m-x}} & \text{when } x<a_m,\\ \dfrac{1}{\sqrt{z-b_m}}=\dfrac{1}{\sqrt{x-b_m}} & \text{when } x>b_m,\\ \dfrac{1}{\sqrt{z-b_m}}=-\dfrac{i}{\sqrt{b_m-x}} & \text{when } x<b_m,\end{array}\right\}\ \begin{array}{l}y=+0,\\ m=1,2,\ldots,n.\end{array} \tag{224}$$

Assuming in (221) $y=+0$ and taking into account (224), we will find

$$F_0(z)=\frac{P_{n-1}(x)}{\sqrt{(x-a_1)(x-a_2)\ldots(x-a_n)(x-b_1)(x-b_2)\ldots(x-b_n)}}$$
$$\text{when } x>b_n,\ y=+0,$$
$$F_0(z)=-\frac{iP_{n-1}(x)}{\sqrt{(x-a_1)(x-a_2)\ldots(x-a_n)(x-b_1)(x-b_2)\ldots(x-b_{n-1})(b_n-x)}}$$
$$\text{when } a_n<x<b_n,\ y=+0,$$
$$F_0(z)=-\frac{P_{n-1}(x)}{\sqrt{(x-a_1)\ldots(x-a_{n-1})(a_n-x)(x-b_1)\ldots(x-b_{n-1})(b_n-x)}}$$
$$\text{when } b_{n-1}<x<a_n,\ y=+0,$$
$$F_0(z)=\frac{iP_{n-1}(x)}{\sqrt{(x-a_1)\ldots(x-a_{n-1})(a_n-x)(x-b_1)\ldots(x-b_{n-2})(b_{n-1}-x)(b_n-x)}}$$
$$\text{when } a_{n-1}<x<b_{n-1},\ y=+0,$$

etc., i.e.,

$$F_0(z)=(-1)^{n-m+1}\frac{iP_{n-1}(x)}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} \tag{225}$$

when $a_m < x < b_m \ (m = 1, 2, \ldots, n)$, $y = +0$,

$$F_0(z) = (-1)^{n-m} \frac{P_{n-1}(x)}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} \tag{226}$$

when $b_m < x < a_{m+1} \ (m = 0, 1, 2, \ldots, n)$, $y = +0$, if one were to take $b_0 = -\infty$, $a_{n+1} = \infty$.

Formulas (225) and (226) show that relations (223) indeed take place.

Let us find, further, according to (221)

$$\begin{aligned} F_0(z) &= \frac{P_{n-1}(z)}{z^n \sqrt{\prod_{m=1}^{n}\left(1-\frac{a_m}{z}\right)\left(1-\frac{b_m}{z}\right)}} = \\ &= \frac{1}{z^n}(c_0 + c_1 z + \ldots + c_{n-1} z^{n-1}) \prod_{m=1}^{n}\left(1+\frac{a_m}{2z}+\frac{3a_m^2}{8z^2}+\ldots\right) \times \\ &\times \left(1+\frac{b_m}{2z}+\frac{3b_m^2}{8z^2}+\ldots\right) = \frac{c_{n-1}}{z}+\frac{\alpha_2}{z^2}+\frac{\alpha_3}{z^3}+\ldots, \end{aligned} \tag{227}$$

where $\alpha_2, \alpha_3, \ldots$ – certain real coefficients. Comparing (227) and (218), we will find that function $F_0(z)$ will satisfy condition (218) if one were to assume

$$c_{n-1} = -P = -\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt; \tag{228}$$

According to (219), (225) and (228) we find

$$p(x) = \frac{(-1)^{n-m+1}(c_0 + c_1 x + \ldots + c_{n-2}x^{n-2} - Px^{n-1})}{\pi\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}},$$
$$a_m < x < b_m \ (m = 1, 2, \ldots, n). \tag{229}$$

Formula (229) gives the solution to equation (209) for the special case when $f'(x) = 0$ when $a_m < x < b_m \ (m = 1, 2, \ldots, n)$. When $n = 1$, (229) turns into formula (147) found earlier for this case.

Passing to the general case, we will look for function $F(z)$, which satisfies conditions (215), (216) and (218), in the form

$$F(z)=\frac{F_0(z)}{P_{n-1}(z)}\Phi(z), \tag{230}$$

reducing the determination of function $F(z)$ to the finding of function $\Phi(z)$.

As can be seen from (225), when $a_m < x < b_m (m=1,2,\ldots,n)$ $y=+0$ function $F_0(z)$ is pure imaginary, polynomial $P_{n-1}(z)$ when $y=+0$ obtains real values. Consequently,

$$\mathrm{Re}\, F(z)=\frac{F_0(z)}{P_{n-1}(z)} i\, \mathrm{Im}\, \Phi(z) \tag{231}$$

when $$a_m < x < b_m (m=1,2,\ldots,n),\ y=+0.$$

Substituting (225) and (215) into (231), we will find that condition (215) for function $F(z)$ will correspond to condition

$$f(x)=\frac{(-1)^{n-m}}{\sqrt{\left|\prod\limits_{m=1}^{n}(x-a_m)(x-b_m)\right|}}\mathrm{Im}\,\Phi(x),$$

i.e., condition

$$\mathrm{Im}\,\Phi(x)=(-1)^{n-m}\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}f(x) \tag{232}$$

when $a_m < x < b_m (m=1,2,\ldots,n)$, $y=+0$ for function $\Phi(z)$.

As can be seen from (226), when $b_m < x < a_{m+1}$ $(m=0,1,2,\ldots,n)$, $y=+0$ function $F_0(z)$ obtains real values. Consequently,

$$\mathrm{Im}\, F(z)=\frac{F_0(z)}{P_{n-1}(z)}\mathrm{Im}\,\Phi(z) \tag{233}$$

when $$b_m < x < a_{m+1}\ (m=0,1,2,\ldots,n),\ y=+0.$$

and condition (216) for function $F(z)$ is reduced to the condition

$$\mathrm{Im}\,\Phi(x)=0 \quad \text{при } b_m < x < a_{m+1}\ (m=0,1,2,\ldots,n),\ y=+0 \tag{234}$$

for function $\Phi(z)$. We will look for function $\Phi(z)$ in the form

$$\Phi(z)=\sum_{m=1}^{n}\int_{a_m}^{b_m}\frac{g(t)\,dt}{t-z}+P_{n-1}(z). \tag{235}$$

According to (216) and (217) we will have

$$\left.\begin{aligned}&\operatorname{Im}\Phi(z)=0\\ &\qquad\text{when } b_m<x<a_{m+1}\ (m=0,1,2,\ldots,n),\ y=\pm 0,\\ &\operatorname{Im}\Phi(z)=\pi g(x)\\ &\qquad\text{when } a_m<x<b_m\quad (m=1,2,\ldots,n),\quad y=+0.\end{aligned}\right\} \tag{236}$$

Comparing (236) with (232) and (234), we will find that function $\Phi(z)$, defined by relation (235), will satisfy conditions (232) and (234), if one were to assume

$$g(x)=\frac{(-1)^{n-m}}{\pi}\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}\,f'(x),\qquad a_m<x<b_m\ (m=1,2,\ldots,n). \tag{237}$$

Substituting (237) into (235), we will find

$$\Phi(z)=\sum_{m=1}^{n}\frac{(-1)^{n-m}}{\pi}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-z}+P_{n-1}(z). \tag{238}$$

Substituting (221) and (238) into (230), we will find

$$F(z)=\frac{\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-z}+P_{n-1}(z)}{\sqrt{\prod_{m=1}^{n}(z-a_m)(z-b_m)}}. \tag{239}$$

Thus, function $F(z)$, defined by relation (239), will satisfy conditions (215) and (216).

We find further

$$F(z)=\frac{1}{z^n}\prod_{m=1}^{n}\left(1+\frac{a_m}{2z}+\frac{3a_m^2}{8z^2}+\dots\right)\left(1+\frac{b_m}{2z}+\frac{3b_m^2}{8z^2}+\dots\right)\times$$
$$\times\left(\frac{b_1}{z}+\frac{b_2}{z^2}+\dots+c_0+c_1z+\dots+c_{n-1}z^{n-1}\right)=$$
$$=\frac{c_{n-1}}{z}+\frac{c_2'}{z^2}+\frac{c_3'}{z^3}+\dots \tag{240}$$

Thus, under condition (228) function $F(z)$, defined by relation (239), will satisfy condition (218) in the neighborhood of the point at infinity.

From (225) we find

$$\frac{1}{\sqrt{\prod_{m=1}^{n}(z-a_m)(z-b_m)}}=\frac{F_0(z)}{P_{n-1}(z)}=\frac{(-1)^{n-l+1}i}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}$$
$$\text{when } a_l<x<b_l\ (l=1,2,\dots,n),\ y=+0. \tag{241}$$

From (239), according to (241) we find

$$\operatorname{Im}F(x)=\frac{(-1)^{n-l+1}}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}\left[\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\times\right.$$
$$\left.\times\lim_{y\to+0}\operatorname{Re}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-z}+P_{n-1}(x)\right] \tag{242}$$
$$\text{when } a_l<x<b_l\ (l=1,2,\dots,n),\ y=+0.$$

But, as we already known from § 3, if function $f'(t)$ at point $t=x$ $(a_l<x<b_l)$ satisfies the condition of Lipschitz, then

$$\lim_{y\to+0}\operatorname{Re}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-z}=$$
$$=\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x}\qquad(m=1,2,\dots,n), \tag{243}$$

where when $m = l$ under the integral standing in the right side of formula (243) should be understood as its principal value. Substituting (243) into (242) and (242) into (219), we will find

$$p(x) = \frac{(-1)^{n-l+1}}{\pi\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} \times$$

$$\times\left[\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x}+P_{n-1}(x)\right], \tag{244}$$

$$a_l < x < b_l \quad (l = 1, 2, \ldots, n),$$

where

$$P_{n-1}(x) = c_0 + c_1 x + \ldots + c_{n-2}x^{n-2} - Px^{n-1}, \tag{245}$$

$$P = \sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt. \tag{246}$$

When $n=1,\ a_1=-a,\ b_1=a$ formula (244) coincides with formula (163) found earlier by us for this special case.

Formula (244) for the unknown function $p(x)$ is found by us from relation (219). Function $F(z)$, which appears in (219), satisfies the boundary condition (215):

$$\operatorname{Re} F(z) = f'(x) \quad \text{when } y = +0,\ a_m < x < b_m \ (m = 1, 2, \ldots, n),$$

where

$$\operatorname{Re} F(z) = \frac{\partial}{\partial x}\sum_{m=1}^{n} V_m(x, 0) = \frac{\partial}{\partial x}\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-x|}\,dt \tag{247}$$

$$\text{when } y = +0$$

according to (213). Thus, the found function $p(x)$ will satisfy the relation

$$\frac{\partial}{\partial x}\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-x|}\,dt = f'(x), \tag{248}$$

$$a_m < x < b_m \quad (m = 1, 2, \ldots, n).$$

However, from relations (248) it does not follow yet that the found function $p(x)$ will satisfy the initial equation (209). Integrating (248) with respect to x, we will find

$$\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-x|}dt = f(x)+\alpha_m, \quad a_m<x<b_m \ (m=1,2,\ldots,n), \tag{249}$$

where $\alpha_1, \alpha_2, \ldots, \alpha_n$ — constants, i.e., by substituting the found function $p(x)$ into the left side of equation (209), we will obtain the function of argument x, which on each of the intervals $a_m<x<b_m$ $(m=1,2,\ldots,n)$ can differ from the assigned function $f(x)$ by a certain constant α_m. However, function $p(x)$, determined by formula (244), contains n arbitrary constants $c_0, c_1, \ldots, c_{n-2}$ and P, which are coefficients of the polynomial $P_{n-1}(x)$. Thus, additional constant components $\alpha_1, \alpha_2, \ldots, \alpha_n$, which we will obtain as a result of the substitution of function $p(x)$ from (244) into the initial equation (209), will be functions of constants $c_0, c_1, \ldots, c_{n-2}, P$, and in this case linear functions, since these constants enter linearly into the expression for function $p(x)$. Equating to zero these additional constant components $\alpha_1, \alpha_2, \ldots, \alpha_n$, we will obtain n linear equations for the determination of constants $c_0, c_1, \ldots, c_{n-2}$ and P, which enter into the expression (244) found by us for function $p(x)$. Having thus determined constants $c_0, c_1, \ldots, c_{n-2}$ and P, we will obtain the solution of the initial equation (209).

In the contact problem of the theory of elasticity we will encounter later the case in which function $f(x)$, which stands in the right side of equation (209), is assigned only to within the arbitrary constant component, common for all intervals $a_m<x<b_m$ $(m=1,2,\ldots,n)$, but then quantity directly is assigned

$$P=\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt.$$

In this case for resolution of the problem it is sufficient to express constants $c_0, c_1, \ldots, c_{n-1}$, which enters into formula (244), in terms of the given quantity P. Below we write out the equations,

which in this case determine constants $c_0, c_1, \dots, c_{n-2}$. From (247) we find

$$\int_{b_m}^{a_{m+1}} [\operatorname{Re} F(z)]_{y=+0}\,dx = \sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-a_{m+1}|}\,dt - \sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-b_m|} = f(a_{m+1}) - f(b_m),\quad m=1,2,\dots,n-1 \tag{250}$$

according to (209). If function $f(x)$ is assigned to with in the constant component, common for all intervals $a_m \leq x \leq b_m$ $(m=1,2,\dots,n)$, the differences standing in right sides of relations (250) will be fully defined.

In accordance with relations (226) from (239) we find

$$[\operatorname{Re}F(z)]_{y=+0} = (-1)^{n-m}\frac{\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x}+P_{n-2}(x)}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} \tag{251}$$

$$\text{при } b_m < x < a_{m+1} \quad (m=1,2,\dots,n-1).$$

Substituting (251) into (250) and taking into account designation (245)

$$P_{n-2}(x) = c_0 + c_1x + \dots + c_{n-3}x^{n-3} - Px^{n-2},$$

we will obtain $n-1$ equation

$$\sum_{i=0}^{n-3} c_i\int_{b_m}^{a_{m+1}}\frac{x^i\,dx}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} = (-1)^{n-m}[f(a_{m+1})-f(b_m)] + P\int_{b_m}^{a_{m+1}}\frac{x^{n-2}\,dx}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} -$$
$$-\frac{1}{\pi}\int_{b_m}^{a_{m+1}}\frac{1}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}\times\left[\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x}\right]dx, \tag{252}$$

$$m=1, 2, \ldots, n-1$$

for the determination of coefficients $c_0, c_1, \ldots, c_{n-2}$.

In the contact of the theory of elasticity we will also encounter the case in which the function $f(x)$ standing in the right side of equation (209) is assigned in each of intervals $a_m \leqslant x \leqslant b_m$. $(m=1, 2, \ldots, n)$ only to within its arbitrary additive constant for each interval, but then n of quantity $P_1, P_2, \ldots, P_n$ is assigned:

$$P_m = \int_{a_m}^{b_m} p(t)\,dt \quad (m=1, 2, \ldots, n). \tag{253}$$

In this case

$$P = \sum_{m=1}^{n} P_m, \tag{254}$$

and we will find coefficients $c_0, c_1, \ldots, c_{n-2}$ according to (244) from equations

$$\begin{aligned} \sum_{i=0}^{n-2} c_i \int_{a_m}^{b_m} \frac{z^i\,dz}{\sqrt{\left|\prod_{m=1}^{n}(z-a_m)(z-b_m)\right|}} &= \\ = (-1)^{n-m+1}\pi P_m + P \int_{a_m}^{b_m} & \frac{z^{n-1}\,dz}{\sqrt{\left|\prod_{m=1}^{n}(z-a_m)(z-b_m)\right|}} - \\ - \frac{1}{\pi} \int_{a_m}^{b_m} & \frac{1}{\sqrt{\left|\prod_{m=1}^{n}(z-a_m)(z-b_m)\right|}} \times \\ \times \left[\sum_{m=1}^{n} (-1)^{n-m} \int_{a_m}^{b_m} \sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|} \frac{f'(t)\,dt}{t-z}\right] dz, & \\ m=2, 3, \ldots, n, & \end{aligned} \tag{255}$$

which are obtained by direct substitution of function $p(x)$ from (244) into the first $n-1$ from relations (253).

In conclusion of this section let us examine examples.

1) $n=2$, $a_1=-b$, $b_1=-a$, $a_2=a$, $b_2=b$, $f(x)=s$ when $-b<x<-a$, $f(x)=s$ when $a<x<b$, s — is an arbitrary conatant.

In this case formula (244) takes the form

$$p(x)=\frac{\pm(c_0-Px)}{\pi\sqrt{(x^2-a^2)(b^2-x^2)}},\quad a<|x|<b, \tag{256}$$

where the plus sign should prevail when $x < 0$, and the minus sign when $x > 0$. Equations (2.52) form $m = 1$ give

$$c_0\int_{-a}^{a}\frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}=-s+P\int_{-a}^{a}\frac{x\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}. \tag{257}$$

Assuming $x = at$ and designating by k the ratio $\frac{a}{b}$, we will find

$$\int_{-a}^{a}\frac{dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}=\frac{1}{b}\int_{-1}^{1}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=$$
$$=\frac{2}{b}\int_{0}^{1}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}=\frac{2}{b}K(k), \tag{258}$$

where

$$K(k)=\int_{0}^{1}\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{259}$$

is the so-called complete elliptic integral of the first kind, for the calculation of which there are tables.

Further,

$$\int_{-a}^{a}\frac{x\,dx}{\sqrt{(a^2-x^2)(b^2-x^2)}}=0, \tag{260}$$

since in this definite integral the integrand is odd.

Substituting (258) and (260) into (257), we will find

$$c_0 = -\frac{bs}{2K(k)}, \quad k = \frac{a}{b}. \tag{261}$$

Formulas (256) and (261) give a solution to the problem.

2) $n=2$, $a_1 = -b$, $b_1 = -a$, $a_2 = a$, $b_2 = b$, $f(x) = \alpha_1$ when $-b < x < -a$, $f(x) = \alpha_2$ when $a < x < b$, α_1 and α_2 — are arbitrary constants,

$$\int_{-b}^{-a} p(t)\,dt = P_1, \quad \int_a^b p(t)\,dt = P_2.$$

In this case for $p(x)$ formula (256) is retained, but the constant c_0 will be determined this time by the first of equations (255), which gives

$$c_0 \int_a^b \frac{dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = -\pi P_2 + P\int_a^b \frac{x\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}}. \tag{262}$$

Considering $x = \sqrt{b^2-(b^2-a^2)t^2}$ and designating by k quantity $\sqrt{1-\frac{a^2}{b^2}}$, we will find

$$\int_a^b \frac{dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \frac{1}{b}\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} = \frac{1}{b}K(k). \tag{263}$$

Assuming $x = \sqrt{\frac{a^2+b^2}{2}+\frac{b^2-a^2}{2}\sin\varphi}$, we will find

$$\int_a^b \frac{x\,dx}{\sqrt{(x^2-a^2)(b^2-x^2)}} = \frac{1}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} d\varphi = \frac{\pi}{2}. \tag{264}$$

Substituting (263) and (264) into (262), we will obtain

$$c_0\frac{K(k)}{b} = -\pi P_2 + \frac{\pi}{2}P = \frac{\pi}{2}(P_1 - P_2),$$

since

Thus,

$$P = P_1 + P_2, \tag{265}$$

$$t_0 = \frac{\pi b}{2K(k)}(P_1 - P_2), \quad k = \sqrt{1 - \frac{a^2}{b^2}}. \tag{266}$$

Substituting (265) and (266) into (256), we finally will find

$$p(x) = \pm \frac{\frac{\pi b}{2K(k)}(P_1 - P_2) - (P_1 + P_2)x}{\pi \sqrt{(x^2 - a^2)(b^2 - x^2)}}, \quad k = \sqrt{1 - \frac{a^2}{b^2}}, \quad a < |x| < b, \tag{267}$$

where the plus sign prevails when $x < 0$ and the minus sign when $x > 0$.

§ 5. Equation of the Periodic Contact Problem

Let us now examine the solution of equation

$$\sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} p(\varphi) \ln \frac{1}{2\left|\sin \frac{\varphi - \vartheta}{2}\right|} d\varphi = f(\vartheta), \; \alpha_m < \vartheta < \beta_m \quad (m = 1, 2, \ldots, n), \tag{268}$$

where $f(\vartheta)$ — function assigned in n intervals of the argument ϑ $\alpha_m < \varphi < \beta_m$ $(m = 1, 2, \ldots, n)$, and $p(\vartheta)$ — is unknown function subject to determination in these n intervals of the argument.[1] We will subsequently assume that

$$0 < \alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \ldots < \alpha_n < \beta_n < 2\pi. \tag{269}$$

Let us examine inside the circle of the unit radius $\xi^2 + \eta^2 = 1$ of two variables ξ and η:

$$V(\xi, \eta) = \sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} p(\varphi) \ln \frac{1}{R} d\varphi,$$

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[1]Keldysh, M. V. and Sedov, L. I., Effective solution to certain boundary value problems for harmonic functions.

where R – distance between point Q with coordinates ξ, η and point Q' on the circle of the unit radius with vectorial angle φ (Fig. 2):

$$R = \sqrt{(\xi - \cos\varphi)^2 + (\eta - \sin\varphi)^2}.$$

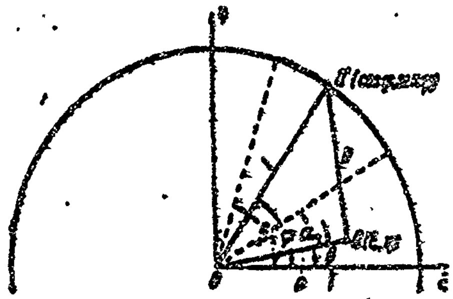

Fig. 2.

Function $V(\xi, \eta)$ constitutes the logarithmic potential of the simple layer with density p, which is located on arcs $(\alpha_1, \beta_1), (\alpha_2, \beta_2), \ldots, (\alpha_n, \beta_n)$ of the unit circle. Just as in § 1, by direct differentiation we will be easily convinced in the fact that function $V(\xi, \eta)$ inside the circle $\xi^2+\eta^2=1$ satisfies the Laplace equation $\frac{\partial^2 V}{\partial \xi^2}+\frac{\partial^2 V}{\partial \eta^2}=0$, it is a harmonic function of variables ξ, η. Passing to polar coordinates ρ, ϑ, i.e., assuming $\xi=\rho\cos\vartheta, \eta=\rho\sin\vartheta$, we will find

$$V = \sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} p(\varphi) \ln \frac{1}{\sqrt{1 - 2\rho\cos(\varphi-\vartheta)+\rho^2}}\, d\varphi. \tag{270}$$

Assuming in (270) $\rho = 1$ and taking into account identity we will find

$$2-2\cos(\varphi-\vartheta) = \left(2\sin\frac{\varphi-\vartheta}{2}\right)^2,$$

$$V = \sum_{m=1}^{n} \int_{\alpha_m}^{\beta_m} p(\varphi) \ln \frac{1}{2\left|\sin\frac{\varphi-\vartheta}{2}\right|}\, d\varphi \quad \text{when } \rho = 1. \tag{271}$$

By comparing (271) and (268), we will arrive at the conclusion that equation (268) is equivalent to condition

$$V = f(\vartheta) \text{ when } \rho = 1, \quad \alpha_m < \vartheta < \beta_m \ (m = 1, 2, \ldots, n), \tag{272}$$

imposed on harmonic function V.

Fulfilling in (270) differentiation with respect to ρ, we will find

$$\frac{\partial V}{\partial \rho} = \sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m} p(\varphi)\frac{\cos(\varphi-\vartheta)-\rho}{1-2\rho\cos(\varphi-\vartheta)+\rho^2}\,d\varphi. \qquad (273)$$

Using the identity

$$\frac{\cos(\varphi-\vartheta)-\rho}{1-2\rho\cos(\varphi-\vartheta)+\rho^2} = \frac{1}{2\rho}\left[\frac{1-\rho^2}{1-2\rho\cos(\varphi-\vartheta)+\rho^2}-1\right],$$

we will be able to give to formula (273) the form

$$\frac{\partial V}{\partial \rho} = \frac{\pi}{\rho}\left[\frac{1}{2\pi}\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m} p(\varphi)\frac{1-\rho^2}{1-2\rho\cos(\varphi-\vartheta)+\rho^2}\,d\varphi-\frac{P}{2\pi}\right], \qquad (274)$$

where

$$P = \sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m} p(\varphi)\,d\varphi. \qquad (275)$$

But expression $\frac{1}{2\pi}\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m} p(\varphi)\frac{1-\rho^2}{1-2\rho\cos(\varphi-\vartheta)+\rho^2}\,d\varphi$ constitutes the Poisson integral

$$\frac{1}{2\pi}\int_0^{2\pi} F(\varphi)\frac{1-\rho^2}{1-2\rho\cos(\varphi-\vartheta)+\rho^2}\,d\varphi,$$

in which $F(\varphi)=p(\varphi)$ when $\alpha_m<\varphi<\beta_m$ $(m=1, 2, \ldots, n)$ and $F(\varphi)=0$ when $\beta_m<\varphi<\alpha_{m+1}$ $(m=0, 1, 2, \ldots, n)$ $(\beta_0=0,\ \alpha_{n+1}=2\pi)$, and approaches $p(\vartheta)$ when $\alpha_m<\vartheta<\beta_m$ $(m=1, 2, \ldots, n)$ and to 0 when $\beta_m<\vartheta<\alpha_{m+1}$ $(m=0, 1, 2, \ldots, n)$, when ρ approaches unity. Thus, from (274) it follows that

$$\frac{\partial V}{\partial \rho} = \pi p(\vartheta)-\frac{P}{2} \quad \text{when } \rho=1,\ \alpha_m<\vartheta<\beta_m\ (m=1, 2, \ldots, n), \qquad (276)$$

$$\frac{\partial V}{\partial \rho} = -\frac{P}{2} \quad \text{when } \rho=1,\ \beta_m<\vartheta<\alpha_{m+1}\ (m=0, 1, 2, \ldots, n). \qquad (277)$$

Thus, having constructed function V, which is harmonic inside the circle $\rho = 1$ and satisfies on this circle the boundary conditions (272) and (277), we will find from (276) the unknown function $p(\phi)$.

The solution of equation (268) is equivalent, thus, to the construction of the function harmonic inside the circle according to mixed boundary conditions assigned on this circle. Just in preceding chapters, we will reduce this boundary value problem to the problem of construction of the function of the complex variable according to boundary conditions assigned for it. Let us consider the function of the complex variable $\zeta = \xi + i\eta$:

$$\Phi(\zeta) = i \sum_{m=1}^{n} \int_{e_m}^{\delta_m} p(\varphi) \frac{\zeta}{e^{i\varphi} - \zeta} d\varphi. \tag{278}$$

Here under the integral sign $\zeta = \xi + i\eta = \rho\cos\theta + i\rho\sin\theta$ — is the complex number depicted in Fig. 2 by point Q, $e^{i\varphi} = \cos\varphi + i\sin\varphi$ — is the complex number depicted on the same figure by point Q'. Let us find

$$\begin{aligned}\frac{\zeta}{e^{i\varphi} - \zeta} &= \frac{\rho\cos\theta + i\rho\sin\theta}{\cos\varphi + i\sin\varphi - \rho\cos\theta - i\rho\sin\theta} = \\ &= \frac{(\rho\cos\theta + i\rho\sin\theta)\left[\cos\varphi - \rho\cos\theta - i(\sin\varphi - \rho\sin\theta)\right]}{\left[\cos\varphi - \rho\cos\theta + i(\sin\varphi - \rho\sin\theta)\right]\left[\cos\varphi - \rho\cos\theta - i(\sin\varphi - \rho\sin\theta)\right]} = \\ &= \frac{\rho\cos(\varphi - \theta) - \rho^2 - i\rho\sin(\varphi - \theta)}{1 - 2\rho\cos(\varphi - \theta) + \rho^2}.\end{aligned} \tag{279}$$

Substituting (279) into (278), we will find

$$\Phi(\zeta) = \sum_{m=1}^{n} \int_{e_m}^{\delta_m} \frac{p(\varphi)\rho\sin(\varphi - \theta)\,d\varphi}{1 - 2\rho\cos(\varphi - \theta) + \rho^2} + i\rho \sum_{m=1}^{n} \int_{e_m}^{\delta_m} \frac{p(\varphi)\left[\cos(\varphi - \theta) - \rho\right]d\varphi}{1 - 2\rho\cos(\varphi - \theta) + \rho^2}. \tag{280}$$

By differentiating with respect to θ from (270) we find

$$\frac{\partial V}{\partial \theta} = \sum_{m=1}^{n} \int_{e_m}^{\delta_m} \frac{p(\varphi)\rho\sin(\varphi - \theta)\,d\varphi}{1 - 2\rho\cos(\varphi - \theta) + \rho^2}. \tag{281}$$

According to (281) and (273), formula (280) can be given the form

$$\Phi(\zeta) = \frac{\partial V}{\partial \theta} + i\rho\frac{\partial V}{\partial \rho}. \tag{282}$$

Thus, the real and imaginary parts of the function of the complex $\Phi(\zeta)$ introduced by us are connected with the harmonic function V by relations

$$\operatorname{Re}\Phi(\zeta)=\frac{\partial V}{\partial\vartheta},\quad \operatorname{Im}\Phi(\zeta)=\rho\frac{\partial V}{\partial\rho}, \tag{283}$$

and conditions (272), (277) and (276) which are satisfied by function V on the unit circle $\rho = 1$, correspond to conditions

$$\operatorname{Re}\Phi(\zeta)=f'(\vartheta)\quad \text{when } \rho=1,\ \alpha_m<\vartheta<\beta_m\ (m=1, 2, \ldots, n), \tag{284}$$
$$\operatorname{Im}\Phi(\zeta)=-\frac{P}{2}\quad \text{when } \rho=1, \beta_m<\vartheta<\alpha_{m+1}\ (m=0, 1, 2, \ldots, n), \tag{285}$$
$$\operatorname{Im}\Phi(\zeta)=\pi p(\vartheta)-\frac{P}{2}\ \text{when } \rho=1, \alpha_m<\vartheta<\beta_m\ (m=1, 2, \ldots, n), \tag{286}$$

which will be satisfied on the unit circle by the function of the complex variable $\phi(\zeta)$. Thus, having constructed function $\phi(\zeta)$, which satisfies conditions (284) and (285), from (286) we will find the unknown function $p(\vartheta)$.

By means of consecutive transformations we find

$$\frac{i\zeta}{e^{i\varphi}-\zeta}=\frac{1}{2}\left(\frac{ie^{i\varphi}+i\zeta}{e^{i\varphi}-\zeta}-i\right)=\frac{1}{2}\left(\frac{ie^{i\frac{\varphi}{2}}+ie^{-i\frac{\varphi}{2}}\zeta}{e^{i\frac{\varphi}{2}}-e^{-i\frac{\varphi}{2}}\zeta}-i\right)=$$
$$=\frac{1}{2}\left[\frac{i\left(\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}\right)+i\left(\cos\frac{\varphi}{2}-i\sin\frac{\varphi}{2}\right)\zeta}{\cos\frac{\varphi}{2}+i\sin\frac{\varphi}{2}-\left(\cos\frac{\varphi}{2}-i\sin\frac{\varphi}{2}\right)\zeta}-i\right]=$$
$$=\frac{1}{2}\left[\frac{\sin\frac{\varphi}{2}(\zeta-1)+i\cos\frac{\varphi}{2}(1+\zeta)}{\cos\frac{\varphi}{2}(1-\zeta)+i\sin\frac{\varphi}{2}(1+\zeta)}-i\right]=\frac{1}{2}\left(\frac{1-i\operatorname{ctg}\frac{\varphi}{2}\frac{1+\zeta}{1-\zeta}}{-\operatorname{ctg}\frac{\varphi}{2}-i\frac{1+\zeta}{1-\zeta}}-i\right)=$$
$$=\frac{1}{2}\left(\frac{1+\operatorname{ctg}^2\frac{\varphi}{2}}{-\operatorname{ctg}\frac{\varphi}{2}-i\frac{1+\zeta}{1-\zeta}}+\operatorname{ctg}\frac{\varphi}{2}-i\right). \tag{287}$$

Substituting (287) into (278), we will obtain

$$\Phi(\zeta)=\frac{1}{2}\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\frac{p(\varphi)\left(1+\operatorname{ctg}^2\frac{\varphi}{2}\right)d\varphi}{-\operatorname{ctg}\frac{\varphi}{2}-i\frac{1+\zeta}{1-\zeta}}+\gamma-i\frac{P}{2}, \tag{288}$$

where

$$\gamma=\frac{1}{2}\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}p(\varphi)\operatorname{ctg}\frac{\varphi}{2}\,d\varphi, \tag{289}$$

and P is determined by relation (275). Assuming further in (288)

we obtain

$$-\operatorname{ctg}\frac{\theta}{2} = t, \tag{290}$$

$$\Phi(\zeta) = \sum_{m=1}^{n}\int_{a_m}^{b_m} \frac{p^0(t)\,dt}{t - i\frac{1+\zeta}{1-\zeta}} + \gamma - i\frac{P}{2}, \tag{291}$$

where

$$a_m = -\operatorname{ctg}\frac{\alpha_m}{2}, \quad b_m = -\operatorname{ctg}\frac{\beta_m}{2} \quad (m = 1, 2, \ldots, n), \tag{292}$$

$$p^0(t) = p(-2\operatorname{arc}\operatorname{ctg} t). \tag{293}$$

According to (269)

$$-\infty < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < \infty. \tag{294}$$

By introducing designation

$$z = i\frac{1+\zeta}{1-\zeta}, \tag{295}$$

we give to formula (291) the form

$$\Phi(\zeta) = F(z) + \gamma - i\frac{P}{2}, \tag{296}$$

where

$$F(z) = \sum_{m=1}^{n}\int_{a_m}^{b_m}\frac{p^0(t)\,dt}{t-z}. \tag{297}$$

Formula (295) determines the complex number z, the real and imaginary part of which are changed when a change in the real and imaginary parts of the complex number ζ. Designating by x and y the real and imaginary parts of the complex number z and assuming in (295) $\zeta = \rho\cos\theta + i\rho\sin\theta$, we will find

$$x + iy = \frac{i(1+\rho\cos\theta + i\rho\sin\theta)}{1-\rho\cos\theta - i\rho\sin\theta} = \frac{[-\rho\sin\theta + i(1+\rho\cos\theta)](1-\rho\cos\theta + i\rho\sin\theta)}{(1-\rho\cos\theta)^2 + \rho^2\sin^2\theta} = \frac{-2\rho\sin\theta + i(1-\rho^2)}{1-2\rho\cos\theta+\rho^2},$$

whence

$$x = \frac{-2\rho\sin\theta}{1-2\rho\cos\theta+\rho^2}, \quad y = \frac{1-\rho^2}{1-2\rho\cos\theta+\rho^2}. \tag{298}$$

From (298) it is clear that $y > 0$ when $\rho < 1$ and $y = 0$ when $\rho = 1$,

i.e., point P with coordinates x, y, which depicts the complex number $z = x + iy$, is in the upper half-plane xOy, when point Q with polar coordinates ρ, θ, which depicts the complex number $\zeta = \rho e^{i\theta}$, is inside the unit circle $\rho = 1$ (Fig. 3), and point P emerges on axis Ox when point Q gets into the circle $\rho = 1$. When $\rho = 1$ formula (298) gives:

$$x = -\frac{\sin\theta}{1-\cos\theta} = -\operatorname{ctg}\frac{\theta}{2}. \tag{299}$$

Relation (299) shows that when point Q describes the circle $\rho = 1$, point P passes axis Ox, where x changes from $-\infty$ to $+\infty$ when θ changes from 0 to 2 π.

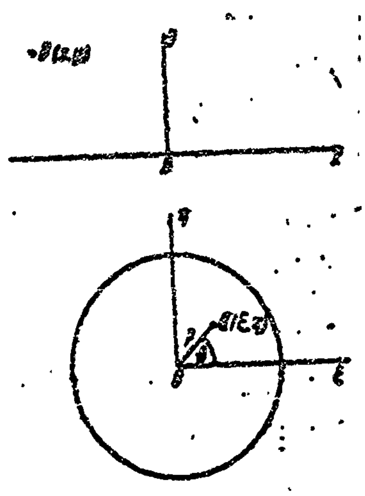

Fig. 3.

From (296) we find

$$\left.\begin{aligned} \operatorname{Re} F(z) &= \operatorname{Re}\Phi(\zeta) - \gamma, \\ \operatorname{Im} F(z) &= \operatorname{Im}\Phi(\zeta) + \frac{P}{2}. \end{aligned}\right\} \tag{300}$$

From (300) it follows that conditions (284) and (285) for function $\Phi(\zeta)$ correspond to the condition

$$\begin{aligned} &[\operatorname{Re} F(z)]_{y=+0} = f^{\circ}(\theta) - \gamma && \text{when } \alpha_m < \theta < \beta_m \ (m = 1, 2, \ldots, n), \\ &[\operatorname{Im} F(z)]_{y=+0} = 0 && \text{when } \beta_m < \theta < \alpha_{m+1} \ (m = 0, 1, 2, \ldots, n) \end{aligned}$$

for function $F(z)$, which according to (299) and (292) can be given the form

$$\operatorname{Re} F(z) = f'(-2 \operatorname{arc\,ctg} z) - \gamma \quad \text{when } y = +0,\ a_m < x < b_m \ (m = 1, 2, \ldots, n), \tag{301}$$

$$\operatorname{Im} F(z) = 0 \quad \text{when } y = +0,\ b_m < x < a_{m+1} \ (m = 0, 1, 2, \ldots, n), \ (b_0 = -\infty,\ a_{n+1} = \infty). \tag{302}$$

From (300) and (286) it follows that

$$\operatorname{Im} F(z) = \pi p(-2 \operatorname{arc\,ctg} x) = \pi p^{*}(x) \text{ when } y = +0,\ a_m < x < b_m \ (m = 1, 2, \ldots, n), \tag{303}$$

if one were to use designation (293).

Relations (302) and (303) directly ensue also from formulas (216) and (217), if one were to compare (297) with (210) and (211).

As showed in § 4, function $F(z)$, determined by formula (239), satisfies conditions (215) and (216). Thus, function

$$F_1(z) = \frac{\frac{1}{\pi} \sum_{m=1}^{n} (-1)^{n-m} \int_{a_m}^{b_m} \sqrt{\left| \prod_{m=1}^{n} (t - a_m)(t - b_m) \right|} \frac{f'(-2 \operatorname{arc\,ctg} t)}{t - z} dt + P_{n-1}(z)}{\sqrt{\prod_{m=1}^{n} (z - a_m)(z - b_m)}} \tag{304}$$

will satisfy conditions

$$\left. \begin{aligned} &\operatorname{Re} F_1(z) = f'(-2 \operatorname{arc\,ctg} x) \\ &\quad \text{when } y = +0,\ a_m < x < b_m \ (m = 1, 2, \ldots, n), \\ &\operatorname{Im} F_1(z) = 0 \\ &\quad \text{when } y = +0,\ b_m < x < a_{m+1} \ (m = 0, 1, 2, \ldots, n). \end{aligned} \right\} \tag{305}$$

We will look for function $F(z)$ in the form of the sum

$$F(z) = F_1(z) + F_2(z). \tag{306}$$

Then according to (301), (302) and (305) function $F_2(x)$ should satisfy the conditions

$$\left.\begin{array}{l}\mathrm{Re}\,F_1(z)=-\gamma \text{ when } y=+0,\ a_m<x<b_m\ (m=1,2,\ldots,n),\\ \mathrm{Im}\,F_2(z)=0 \text{ when } y=+0,\ b_m<x<a_{m+1}\ (m=0,1,\ldots,n).\end{array}\right\} \qquad (307)$$

Furthermore, as can be seen from (297) and (304), $F(\infty)=F_1(\infty)=0$. Consequently, function $F(z)$ should also satisfy the condition

$$F_2(\infty)=0. \qquad (308)$$

Let us examine function

$$\sqrt{\prod_{m=1}^{n}\frac{z-b_m}{z-a_m}}.$$

According to (224)

$$\left.\begin{array}{ll} & \sqrt{\prod\limits_{m=1}^{n}\frac{z-b_m}{z-a_m}}=\sqrt{\left|\prod\limits_{m=1}^{n}\frac{x-b_m}{x-a_m}\right|}\\ \text{when} & y=+0,\ b_m<x<a_{m+1}\ (m=0,1,\ldots,n),\\ & \sqrt{\prod\limits_{m=1}^{n}\frac{z-b_m}{z-a_m}}=i\sqrt{\left|\prod\limits_{m=1}^{n}\frac{x-b_m}{x-a_m}\right|}\\ \text{when} & y=+0,\ a_m<x<b_m\ (m=1,2,\ldots,n)\end{array}\right\} \qquad (309)$$

and

$$\sqrt{\prod_{m=1}^{n}\frac{z-b_n}{z-a_n}}\to+1, \text{ when } z\to\infty. \qquad (310)$$

According to relations (309) and (310) function

$$F_2(z)=\gamma\left[\sqrt{\prod_{m=1}^{n}\frac{z-b_m}{z-a_m}}-1\right] \qquad (311)$$

satisfies both conditions (307) and condition (308).

Substituting (304) and (311) into (306), we will find

$$F(z)=\frac{\frac{1}{\pi}\sum\limits_{m=1}^{n}(-1)^{n-m}\int\limits_{a_m}^{b_m}\sqrt{\left|\prod\limits_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f(\xi-2\,\mathrm{arcctg}\,\xi)}{t-z}\,dt+P_n^{0}(z)}{\sqrt{\prod\limits_{m=1}^{n}(z-a_m)(z-b_m)}}-\gamma. \qquad (312)$$

where $P_n^{**}(z)$ is a polynomial in z of the power n:

$$P_n^{\circ}(z) = c_0^{\circ} + c_1^{\circ} z + c_2^{\circ} z^2 + \ldots + c_{n-1}^{\circ} z^{n-1} + \gamma z^n. \tag{313}$$

Using identity

$$\frac{1}{t-z} = \frac{1}{t-z}\left[\prod_{m=1}^{n} \frac{z-b_m}{t-b_m} + 1 - \prod_{m=1}^{n}\left(1 - \frac{t-z}{t-b_m}\right)\right] =$$
$$= \frac{1}{t-z} \prod_{m=1}^{n} \frac{z-b_m}{t-b_m} + \varphi_0(t) + \varphi_1(t) z + \ldots + \varphi_{n-2}(t) z^{n-2},$$

where $\varphi_0(t), \varphi_1(t), \ldots, \varphi_{n-2}(t)$ — functions of argument t, the evident expression of which we do not write out, it is possible to give (312) formula the form

$$F(z) = \frac{\sum_{m=1}^{n} (-1)^{n-m}}{\pi \sqrt{\prod_{m=1}^{n} (z-a_m)(z-b_m)}} \times$$
$$\times \int_{a_m}^{b_m} \sqrt{\left|\prod_{m=1}^{n} (t-a_m)(t-b_m)\right|} \cdot \prod_{m=1}^{n} \frac{z-b_m}{t-b_m} \frac{f(-2\operatorname{arcctg} t)}{t-z}\, dt +$$
$$+ \frac{P_n^{**}(z)}{\sqrt{\prod_{m=1}^{n} (z-a_m)(z-b_m)}} - \gamma_0$$

or

$$F(z) = -\frac{1}{\pi} \sqrt{\prod_{m=1}^{n} \frac{z-b_m}{z-a_m}} \sum_{m=1}^{n} \int_{a_m}^{b_m} \sqrt{\left|\prod_{m=1}^{n} \frac{t-a_m}{t-b_m}\right|} \frac{f(-2\operatorname{arcctg} t)}{t-z}\, dt +$$
$$+ \frac{P_n^{**}(z)}{\sqrt{\prod_{m=1}^{n} (z-a_m)(z-b_m)}} - \gamma_0. \tag{314}$$

since

$$\prod_{m=1}^{n} (t-b_m) = (-1)^{n-m+1} \left|\prod_{m=1}^{n} (t-b_m)\right|$$

when $a_m < t < b_m \quad (m = 1, 2, \ldots, n)$.

Here $P_n^{**}(z)$ is a polynomial in z of n power, the coefficients of which, with the exception of the latter are different from coefficients of the polynomial $P_n^*(z)$.

Using, further, the identity

$$\frac{1}{t-z}=\frac{1+tz}{(1+t^2)(t-z)}+\frac{t}{1+t^2},$$

let us give to formula (314) the final form

$$F(z)=-\frac{1}{\pi}\sqrt{\prod_{m=1}^{n}\frac{z-b_m}{z-a_m}}\times$$
$$\times\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{t-a_m}{t-b_m}\right|}\frac{f'(-2\operatorname{arcctg} t)(1+tz)}{(1+t^2)(t-z)}dt+$$
$$+\frac{P_n(z)}{\sqrt{\prod_{m=1}^{n}(z-a_m)(z-b_m)}}-\gamma_0 \qquad (315)$$

where

$$P_n(z)=c_0+c_1z+\ldots+c_nz^n \qquad (316)$$

is a polynomial with certain new coefficients $c_0, c_1, \ldots, c_n$. Substituting (315) into (303), let us find in accordance with (309) and (224):

$$p(-2\operatorname{arcctg} x)=-\frac{1}{\pi^2}\sqrt{\left|\prod_{m=1}^{n}\frac{x-b_m}{x-a_m}\right|}\times$$
$$\times\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{t-a_m}{t-b_m}\right|}\frac{f'(-2\operatorname{arcctg} t)(1+tx)}{(1+t^2)(t-x)}dt+$$
$$+\frac{(-1)^{n-m+1}P_n(x)}{\pi\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} \qquad (317)$$

$$\text{when } a_m<x<b_m \quad (m=1,2,\ldots,n).$$

Substituting into (317) a_m and b_m from (292) and assuming in (317)

$$t=-\operatorname{ctg}\frac{\varphi}{2},\quad x=-\operatorname{ctg}\frac{\theta}{2},$$

we will obtain

$$p(\vartheta)=-\frac{1}{2\pi^2}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\vartheta-\alpha_m}{2}}\right|}\times$$

$$\times\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi+$$

$$+\frac{(-1)^{n-m+1}\sqrt{\prod_{m=1}^{n}\sin\frac{\alpha_m}{2}\sin\frac{\beta_m}{2}}\left(\sin\frac{\vartheta}{2}\right)^n P_n\left(-\operatorname{ctg}\frac{\vartheta}{2}\right)}{\pi\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\vartheta-\alpha_m}{2}\sin\frac{\vartheta-\beta_m}{2}\right|}},$$

since

$$-\operatorname{ctg}\frac{\vartheta}{2}+\operatorname{ctg}\frac{\alpha_m}{2}=\frac{\sin\frac{\vartheta-\alpha_m}{2}}{\sin\frac{\alpha_m}{2}\sin\frac{\vartheta}{2}},$$

$$-\operatorname{ctg}\frac{\vartheta}{2}+\operatorname{ctg}\frac{\beta_m}{2}=\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\beta_m}{2}\sin\frac{\vartheta}{2}},\quad\frac{1+\operatorname{ctg}\frac{\varphi}{2}\operatorname{ctg}\frac{\vartheta}{2}}{-\operatorname{ctg}\frac{\varphi}{2}+\operatorname{ctg}\frac{\vartheta}{2}}=\operatorname{ctg}\frac{\varphi-\vartheta}{2},$$

or finally

$$p(\vartheta)=-\frac{1}{2\pi^2}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\vartheta-\alpha_m}{2}}\right|}\times$$

$$\times\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi+$$

$$+(-1)^{n-m+1}\frac{\sum_{m=0}^{n}\gamma_m\sin^{n-m}\frac{\vartheta}{2}\cos^m\frac{\vartheta}{2}}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\vartheta-\alpha_m}{2}\sin\frac{\vartheta-\beta_m}{2}\right|}}\qquad(318)$$

when $\alpha_m<\vartheta<\beta_m\quad(m=1,2,\ldots,n)$,

or finally

$$\gamma_m=\frac{1}{\pi}(-1)^m\sqrt{\prod_{m=1}^{n}\sin\frac{\alpha_m}{2}\sin\frac{\beta_m}{2}}c_m\quad(m=0,1,2,\ldots,n).\qquad(319)$$

Let us now turn to the determination of coefficients $\gamma_0, \gamma_1, \ldots, \gamma_n$, which enter into formula (318) for the unknown function $F(0)$. From (278) it is clear that $\Phi(0) = 0$. Further from (295) it follows that $z = i$ when $\zeta = 0$. On the basis of this, formula (296) gives

$$F(i)+\gamma=i\frac{P}{2}. \tag{320}$$

Substituting (292) into (315), assuming in (315) $t=-\operatorname{ctg}\frac{\varphi}{2}$; $z=i$, and taking into account the identity

$$i+\operatorname{ctg}\frac{\alpha_m}{2}=\frac{\cos\frac{\alpha_m}{2}+i\sin\frac{\alpha_m}{2}}{\sin\frac{\alpha_m}{2}}=\frac{e^{i\frac{\alpha_m}{2}}}{\sin\frac{\alpha_m}{2}},\quad i+\operatorname{ctg}\frac{\beta_m}{2}=\frac{e^{i\frac{\beta_m}{2}}}{\sin\frac{\beta_m}{2}},$$

$$\frac{1-i\operatorname{ctg}\frac{\varphi}{2}}{-\operatorname{ctg}\frac{\varphi}{2}-i}=\frac{\left(1-i\operatorname{ctg}\frac{\varphi}{2}\right)\left(-\operatorname{ctg}\frac{\varphi}{2}+i\right)}{\operatorname{ctg}^2\frac{\varphi}{2}+1}=i,$$

we will find

$$F(i)=-\frac{i}{2\pi}e^{\frac{i}{4}\sum_{m=1}^{n}(\beta_m-\alpha_m)}\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\,d\varphi+$$

$$+\frac{\sqrt{\prod_{m=1}^{n}\sin\frac{\alpha_m}{2}\sin\frac{\beta_m}{2}}\sum_{m=0}^{n}c_m i^m}{e^{\frac{i}{4}\sum_{m=1}^{n}(\alpha_m+\beta_m)}}-\gamma. \tag{321}$$

Substituting (321) into (320) and taking into account (319), we will obtain equation

$$\sum_{m=0}^{n}(-i)^m\gamma_m=i\frac{P}{2}\left[\cos\frac{1}{4}\sum_{m=1}^{n}(\alpha_m+\beta_m)+i\sin\frac{1}{4}\sum_{m=1}^{n}(\alpha_m+\beta_m)\right]+$$

$$+\frac{i}{2\pi}\left(\cos\frac{1}{2}\sum_{m=1}^{n}\beta_m+i\sin\frac{1}{2}\sum_{m=1}^{n}\beta_m\right)\times$$

$$\times\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\,d\varphi,$$

whence

$$
\left.\begin{aligned}
&-\gamma_2+\gamma_4-\gamma_6+\gamma_8-\ldots=\frac{P}{2\pi}\sin\frac{1}{4}\sum_{m=1}^{n}(\alpha_m+\beta_m)+\\
&+\frac{1}{2\pi^2}\left(\sin\frac{1}{2}\sum_{m=1}^{n}\beta_m\right)\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\,d\varphi,\\
&-\gamma_1+\gamma_3-\gamma_5+\gamma_7-\ldots=\frac{P}{2\pi}\cos\frac{1}{4}\sum_{m=1}^{n}(\alpha_m+\beta_m)+\\
&+\frac{1}{2\pi^2}\left(\cos\frac{1}{2}\sum_{m=1}^{n}\beta_m\right)\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\,d\varphi.
\end{aligned}\right\} \tag{322}
$$

Further from (283) and (272) it follows that

$$
\int_{\beta_k}^{\alpha_{k+1}}[\operatorname{Re}\Phi(\zeta)]_{\rho=1}\,d\theta=(V)_{\substack{\rho=1\\ \theta=\alpha_{k+1}}}-(V)_{\substack{\rho=1\\ \theta=\beta_k}}=f(\alpha_{k+1})-f(\beta_k),
\quad k=1,\,2,\ldots,\,n-1. \tag{323}
$$

If function $f(\theta)$ is assigned to within the arbitrary constant component common for all intervals $(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n)$, the right sides of equations (323) will have fully defined values. From (296) we find

$$
[\operatorname{Re}F(\zeta)]_{\rho=1}=[\operatorname{Re}F(z)]_{y\to+0}+\gamma. \tag{324}
$$

Substituting (315) into (324), let us find in accordance with (309) and (224)

$$
\begin{aligned}
[\operatorname{Re}F(\zeta)]_{\rho=1}=-\frac{1}{\pi}\sqrt{\left|\prod_{m=1}^{n}\frac{x-b_m}{x-a_m}\right|}\times\\
\times\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{t-a_m}{t-b_m}\right|}\frac{f(-2\operatorname{arcctg}t)\,(1+tx)}{(1+t^2)(t-x)}\,dt+\\
+\frac{(-1)^{n-k}P_n(x)}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}
\end{aligned} \tag{325}
$$

$$
\text{when } b_k<x<a_{k+1}\quad(k=1,\,2,\ldots,\,n-1).
$$

Substituting (292) into (325), assuming in (325) $t=-\operatorname{ctg}\frac{\varphi}{2}$, $x=-\operatorname{ctg}\frac{\theta}{2}$

and taking into account (319), we will find

$$[\operatorname{Re} F(\zeta)]_{r=1} = -\frac{1}{2\pi}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\vartheta-\alpha_m}{2}}\right|}\times$$

$$\times\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi+$$

$$+\frac{n(-1)^{n-k}\sum_{m=0}^{n}\gamma_m\sin^{n-m}\frac{\vartheta}{2}\cos^{m}\frac{\vartheta}{2}}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\vartheta-\alpha_m}{2}\sin\frac{\vartheta-\beta_m}{2}\right|}} \tag{326}$$

$$\text{when } \beta_k<\vartheta<\alpha_{k+1}\quad(k=1,2,\ldots,n-1).$$

Substituting (326) into (323), we will obtain the equation

$$\sum_{m=0}^{n}\gamma_m\int_{\beta_k}^{\alpha_{k+1}}\frac{\sin^{n-m}\frac{\vartheta}{2}\cos^{m}\frac{\vartheta}{2}d\vartheta}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\vartheta-\alpha_m}{2}\sin\frac{\vartheta-\beta_m}{2}\right|}}=$$

$$=(-1)^{n-k}\left[\frac{f(\alpha_{k+1})-f(\beta_k)}{\pi}+\frac{1}{2\pi^2}\int_{\beta_k}^{\alpha_{k+1}}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\vartheta-\alpha_m}{2}}\right|}\times\right.$$

$$\left.\times\left(\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi\right)d\vartheta\right] \tag{327}$$

$$(k=1,2,\ldots,n-1),$$

which jointly with equations (322) will form the system of $n+1$ equation for the determination of $n+1$ coefficient $\gamma_0,\gamma_1,\ldots,\gamma_n$.

If function $f(\vartheta)$ is assigned only to within the arbitrary component, its own for each of the intervals $(\alpha_1,\beta_1),(\alpha_2,\beta_2),\ldots,(\alpha_n,\beta_n)$, but then all n quantities are assigned

$$P_k=\int_{\alpha_k}^{\beta_k}p(\vartheta)d\vartheta\quad(k=1,2,\ldots,n), \tag{328}$$

and not just their sum P, then, substituting (318) into (328), we will obtain equations

$$\sum_{m=0}^{n}\gamma_m\int_{\alpha_k}^{\beta_k}\frac{\sin^{n-m}\frac{\theta}{2}\cos^m\frac{\theta}{2}\,d\theta}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\theta-\alpha_m}{2}\sin\frac{\theta-\beta_m}{2}\right|}}=$$
$$=(-1)^{n-k+1}\left[P_k+\frac{1}{2\pi^2}\int_{\alpha_k}^{\beta_k}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\theta-\beta_m}{2}}{\sin\frac{\theta-\alpha_m}{2}}\right|}\times\right.$$
$$\left.\times\left(\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'(\varphi)\operatorname{ctg}\frac{\varphi-\theta}{2}\,d\varphi\right)d\theta\right] \tag{329}$$
$$(k=1,\ 2,\ \ldots,\ n-1),$$

which together with equations (322) will form for this case the system of n + 1 equation for the determination of n + 1 coefficient $\gamma_0, \gamma_1, \ldots, \gamma_n$.

After coefficients $\gamma_0, \gamma_1, \ldots, \gamma_n$ are calculated, formula (318) gives the solution of the initial equation (268).

In the special case when n = 1, $\alpha_1=\pi-\alpha,\ \beta_1=\pi+\alpha$, formulas (318) and (322) take the form

$$p(\theta)=-\frac{1}{2\pi^2}\sqrt{-\frac{\cos\frac{\theta-\alpha}{2}}{\cos\frac{\theta+\alpha}{2}}}\int_{\pi-\alpha}^{\pi+\alpha}\sqrt{-\frac{\cos\frac{\varphi+\alpha}{2}}{\cos\frac{\varphi-\alpha}{2}}}f'(\varphi)\operatorname{ctg}\frac{\varphi-\theta}{2}\,d\varphi-$$
$$-\frac{\gamma_0\sin\frac{\theta}{2}+\gamma_1\cos\frac{\theta}{2}}{\sqrt{-\cos\frac{\theta+\alpha}{2}\cos\frac{\theta-\alpha}{2}}},\qquad \pi-\alpha<\theta<\pi+\alpha, \tag{330}$$

$$\left.\begin{aligned}-\gamma_0&=\frac{P}{2\pi}+\frac{1}{2\pi^2}\cos\frac{\alpha}{2}\int_{\pi-\alpha}^{\pi+\alpha}\sqrt{-\frac{\cos\frac{\varphi+\alpha}{2}}{\cos\frac{\varphi-\alpha}{2}}}f'(\varphi)\,d\varphi,\\ -\gamma_1&=-\frac{1}{2\pi^2}\sin\frac{\alpha}{2}\int_{\pi-\alpha}^{\pi+\alpha}\sqrt{-\frac{\cos\frac{\varphi+\alpha}{2}}{\cos\frac{\varphi-\alpha}{2}}}f'(\varphi)\,d\varphi,\end{aligned}\right\} \tag{331}$$

since when $\pi-\alpha<\theta<\pi+\alpha,\ \alpha<\pi$ we have

$$-\pi<\pi-2\alpha<\vartheta-\alpha<\pi,\quad \pi<\vartheta+\alpha<\pi+2\alpha<3\pi,$$

and consequently,

$$\cos\frac{\vartheta-\alpha}{2}>0,\quad \cos\frac{\vartheta+\alpha}{2}<0. \tag{332}$$

Substituting (331) into (330), we will obtain the formula

$$p(\vartheta)=-\frac{1}{2\pi^2\sqrt{-\cos\frac{\vartheta-\alpha}{2}\cos\frac{\vartheta+\alpha}{2}}}\left[\int_{\pi-\alpha}^{\pi+\alpha}\sqrt{-\frac{\cos\frac{\varphi+\alpha}{2}}{\cos\frac{\varphi-\alpha}{2}}}\times\right.$$
$$\left.\times f'(\varphi)\left(\cos\frac{\vartheta-\alpha}{2}\operatorname{ctg}\frac{\varphi-\vartheta}{2}-\sin\frac{\vartheta-\alpha}{2}\right)d\varphi-\pi P\sin\frac{\vartheta}{2}\right],$$

or, if one were to consider the identity

$$\cos\frac{\vartheta-\alpha}{2}\operatorname{ctg}\frac{\varphi-\vartheta}{2}-\sin\frac{\vartheta-\alpha}{2}=\frac{\cos\frac{\varphi-\alpha}{2}}{\sin\frac{\varphi-\vartheta}{2}},$$

we will obtain

$$p(\vartheta)=-\frac{1}{2\pi^2\sqrt{-\cos\frac{\vartheta-\alpha}{2}\cos\frac{\vartheta+\alpha}{2}}}\times$$
$$\times\left[\int_{\pi-\alpha}^{\pi+\alpha}\frac{\sqrt{-\cos\frac{\varphi-\alpha}{2}\cos\frac{\varphi+\alpha}{2}}\,f'(\varphi)\,d\varphi}{\sin\frac{\varphi-\vartheta}{2}}-\pi P\sin\frac{\vartheta}{2}\right], \tag{333}$$
$$\pi-\alpha<\vartheta<\pi+\alpha.$$

Assuming in (333) $\pi+\phi$ instead of ϕ and $\pi+\vartheta$ instead of ϑ, and taking into account the identity

$$\sin\frac{\alpha-\vartheta}{2}\sin\frac{\alpha+\vartheta}{2}=\frac{1}{2}(\cos\vartheta-\cos\alpha),$$

it is possible to give to formula (333) a somewhat different form

$$p(\pi+\vartheta)=-\frac{\sqrt{2}}{2\pi^2\sqrt{\cos\vartheta-\cos\alpha}}\times$$
$$\times\left[\frac{1}{\sqrt{2}}\int_{-\alpha}^{\alpha}\frac{\sqrt{\cos\varphi-\cos\alpha}\,f'(\pi+\varphi)\,d\varphi}{\sin\frac{\varphi-\vartheta}{2}}-\pi P\cos\frac{\vartheta}{2}\right], \tag{334}$$
$$-\alpha<\vartheta<\alpha.$$

In the maximum case when $\alpha = \pi$, formula (333) gives

$$p(\theta) = -\frac{1}{2\pi^2 \sin\frac{\theta}{2}} \left[\int_0^{2\pi} \frac{\sin\frac{\varphi}{2} f'(\varphi)\, d\varphi}{\sin\frac{\varphi-\theta}{2}} - \pi P \sin\frac{\theta}{2} \right]. \qquad (335)$$

Using the identity

$$\frac{\sin\frac{\varphi}{2}}{\sin\frac{\varphi-\theta}{2}} = \sin\frac{\theta}{2} \operatorname{ctg}\frac{\varphi-\theta}{2} + \cos\frac{\theta}{2},$$

it is possible to give to formula (335) the form

$$p(\theta) = -\frac{1}{2\pi^2} \int_0^{2\pi} f'(\varphi) \operatorname{ctg}\frac{\varphi-\theta}{2}\, d\varphi - \frac{f(2\pi)-f(0)}{2\pi^2} \operatorname{ctg}\frac{\theta}{2} + \frac{P}{2\pi}. \qquad (336)$$

In particular, if

$$f(2\pi) - f(0) = 0,$$

$$p(\theta) = -\frac{1}{2\pi^2} \int_0^{2\pi} f'(\varphi) \operatorname{ctg}\frac{\varphi-\theta}{2}\, d\varphi + \frac{P}{2\pi},$$

or, according to (109),

$$p(\theta) = \frac{1}{\pi} \left[\overline{f'(\varphi)} + \frac{P}{2} \right], \qquad (337)$$

where $\overline{f'(\varphi)}$ is a function conjugate with function $f'(\phi)$.

In conclusion of this section let us examine examples.

1) $n = 1$, $\alpha_1 = \pi - \alpha$, $\beta_1 = \pi + \alpha$, $f(\varphi) = \text{const.}$ when $\pi - \alpha < \varphi < \pi + \alpha$.

In this case formula (334) directly gives

$$p(\pi + \theta) = \frac{P \sqrt{2} \cos\frac{\theta}{2}}{2\pi \sqrt{\cos\theta - \cos\alpha}}. \qquad (338)$$

2) $n = 1$, $\alpha_1 = \pi - \alpha$, $\beta_1 = \pi + \alpha$, $f(\varphi) = -A \cos^2\frac{\varphi}{2} + \text{const.}$ $\pi - \alpha < \varphi < \pi + \alpha$.

In this case

$$f'(\pi+\varphi) = -\frac{A}{2}\sin\varphi \quad \text{when} \quad -\alpha<\varphi<\alpha. \qquad (339)$$

Substituting (339) into (334), we will find

$$p(\pi+\vartheta) = \frac{\sqrt{2}}{2\pi^2\sqrt{\cos\vartheta-\cos\alpha}}\times$$
$$\times\left(\frac{A}{2\sqrt{2}}\int_{-\alpha}^{\alpha}\frac{\sqrt{\cos\varphi-\cos\alpha}\,\sin\varphi\,d\varphi}{\sin\frac{\varphi-\vartheta}{2}}+\pi P\cos\frac{\vartheta}{2}\right), \qquad (340)$$
$$-\alpha<\vartheta<\alpha.$$

Assuming

$$\operatorname{tg}\frac{\varphi}{2} = \operatorname{tg}\frac{\alpha}{2}\sin t,$$

we will find

$$\frac{\sqrt{\cos\varphi-\cos\alpha}}{\sin\frac{\varphi-\vartheta}{2}} = \frac{\sqrt{2\left(\cos^2\frac{\varphi}{2}-\cos^2\frac{\alpha}{2}\right)}}{\sin\frac{\varphi}{2}\cos\frac{\vartheta}{2}-\cos\frac{\varphi}{2}\sin\frac{\vartheta}{2}} =$$
$$= \frac{\sqrt{2}\cos\frac{\alpha}{2}\sqrt{\sec^2\frac{\alpha}{2}-\sec^2\frac{\varphi}{2}}}{\cos\frac{\vartheta}{2}\left(\operatorname{tg}\frac{\varphi}{2}-\operatorname{tg}\frac{\vartheta}{2}\right)} = \frac{\sqrt{2}\sin\frac{\alpha}{2}\cos t}{\cos\frac{\vartheta}{2}\left(\operatorname{tg}\frac{\alpha}{2}\sin t-\operatorname{tg}\frac{\vartheta}{2}\right)},$$
$$\sin\varphi\,d\varphi = \sin\varphi\,\frac{2\operatorname{tg}\frac{\alpha}{2}\cos t}{\sec^2\frac{\varphi}{2}}\,dt = \frac{4\operatorname{tg}\frac{\varphi}{2}\operatorname{tg}\frac{\alpha}{2}\cos t}{\sec^4\frac{\varphi}{2}}\,dt =$$
$$= \frac{4\operatorname{tg}^2\frac{\alpha}{2}\sin t\cos t}{\left(1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t\right)^2}\,dt,$$

and

$$p(\pi+\vartheta) = \frac{\sqrt{2}}{2\pi^2\sqrt{\cos\vartheta-\cos\alpha}}\times$$
$$\times\left[\frac{2A\sin\frac{\alpha}{2}\operatorname{tg}^2\frac{\alpha}{2}}{\cos\frac{\vartheta}{2}}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin t\cos^2 t\,dt}{\left(\operatorname{tg}\frac{\alpha}{2}\sin t-\operatorname{tg}\frac{\vartheta}{2}\right)\left(1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t\right)^2}+\pi P\cos\frac{\vartheta}{2}\right]. \qquad (341)$$

The integrand in (341) can be presented in the form

$$\frac{\sin t\cos^2 t}{\left(\operatorname{tg}\frac{\alpha}{2}\sin t-\operatorname{tg}\frac{\theta}{2}\right)\left(1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t\right)^2}=$$

$$=\frac{c_1}{\operatorname{tg}\frac{\alpha}{2}\sin t-\operatorname{tg}\frac{\theta}{2}}+\frac{c_2+c_3\sin t}{1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t}+\frac{c_4+c_5\sin t}{\left(1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t\right)^2}, \tag{342}$$

where c_1, c_2, c_3, c_4 and c_5 do not depend on t and must satisfy equations

$$\left.\begin{aligned}
c_1-c_3\operatorname{tg}\frac{\theta}{2}-c_5\operatorname{tg}\frac{\theta}{2}&=0,\\
c_3\operatorname{tg}\frac{\alpha}{2}-c_2\operatorname{tg}\frac{\theta}{2}+c_5\operatorname{tg}\frac{\alpha}{2}-c_4\operatorname{tg}\frac{\theta}{2}&=1,\\
2c_1\operatorname{tg}^2\frac{\alpha}{2}-c_3\operatorname{tg}\frac{\theta}{2}\operatorname{tg}^2\frac{\alpha}{2}+c_2\operatorname{tg}\frac{\alpha}{2}+c_4\operatorname{tg}\frac{\alpha}{2}&=0,\\
c_3\operatorname{tg}^3\frac{\alpha}{2}-c_2\operatorname{tg}\frac{\theta}{2}\operatorname{tg}^2\frac{\alpha}{2}&=-1,\\
c_1\operatorname{tg}^4\frac{\alpha}{2}+c_2\operatorname{tg}^3\frac{\alpha}{2}&=0.
\end{aligned}\right\} \tag{343}$$

Substituting (342) into (341), we will find

$$p(\pi+\theta)=\frac{\sqrt{2}}{2\pi^2\sqrt{\cos\theta-\cos\alpha}}\times$$

$$\times\left[\frac{2A\sin\frac{\alpha}{2}\operatorname{tg}^3\frac{\alpha}{2}}{\cos\frac{\theta}{2}}\left(c_1J_1+c_2J_2+c_3J_3+c_4J_4+c_5J_5\right)+\pi P\cos\frac{\theta}{2}\right], \tag{344}$$

where

$$J_1=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dt}{\operatorname{tg}\frac{\alpha}{2}\sin t-\operatorname{tg}\frac{\theta}{2}},\quad J_2=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dt}{1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t},$$

$$J_3=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin t\,dt}{1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t},\quad J_4=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{dt}{\left(1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t\right)^2},$$

$$J_5=\int\limits_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin t\,dt}{\left(1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t\right)^2}.$$

Let us find

$$J_3 = J_4 = 0, \tag{345}$$

since in these definite integrals of the integrands are odd.

Assuming further

$$\operatorname{tg} t = \cos\frac{\alpha}{2}\operatorname{tg}\varphi,$$

we will find

$$dt = \frac{\cos\frac{\alpha}{2}\sec^2\varphi\, d\varphi}{1+\cos^2\frac{\alpha}{2}\operatorname{tg}^2\varphi},$$

$$1+\operatorname{tg}^2\frac{\alpha}{2}\sin^2 t = 1+\frac{\operatorname{tg}^2\frac{\alpha}{2}\operatorname{tg}^2 t}{1+\operatorname{tg}^2 t} = \frac{1+\sec^2\frac{\alpha}{2}\operatorname{tg}^2 t}{1+\operatorname{tg}^2 t} = \frac{\sec^2\varphi}{1+\cos^2\frac{\alpha}{2}\operatorname{tg}^2\varphi},$$

$$J_5 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\frac{\alpha}{2}\, d\varphi = \pi\cos\frac{\alpha}{2},$$

$$J_6 = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos\frac{\alpha}{2}\cos^2\varphi\left(1+\cos^2\frac{\alpha}{2}\operatorname{tg}^2\varphi\right)d\varphi =$$

$$\frac{1}{2}\cos\frac{\alpha}{2}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\left[1+\cos 2\varphi+\cos^2\frac{\alpha}{2}(1-\cos 2\varphi)\right]d\varphi =$$

$$= \frac{\pi}{2}\cos\frac{\alpha}{2}\left(1+\cos^2\frac{\alpha}{2}\right). \tag{346}$$

Assuming

$$\operatorname{tg}\frac{t}{2} = \xi, \quad \operatorname{tg}\frac{\theta}{2} = 2\operatorname{tg}\frac{\alpha}{2}\,\frac{x}{1+x^2} \qquad (-1 < x < 1 \text{ when } -\alpha < \theta < \alpha),$$

we will find

$$dt = \frac{2d\xi}{1+\xi^2},$$

$$\operatorname{tg}\frac{\alpha}{2}\sin t - \operatorname{tg}\frac{\theta}{2} = 2\operatorname{tg}\frac{\alpha}{2}\left(\frac{\xi}{1+\xi^2}-\frac{x}{1+x^2}\right) = \frac{2\operatorname{tg}\frac{\alpha}{2}(\xi-x)(1-\xi x)}{(1+\xi^2)(1+x^2)},$$

$$J_1 = \operatorname{ctg}\frac{\alpha}{2}(1+x^2)\int_{-1}^{1}\frac{d\xi}{(\xi-x)(1-\xi x)} =$$

$$= \operatorname{ctg}\frac{\alpha}{2}\,\frac{1+x^2}{1-x^2}\left(\int_{-1}^{1}\frac{d\xi}{\xi-x}-\int_{-1}^{1}\frac{d\xi}{\xi-\frac{1}{x}}\right), \tag{347}$$

$$-1 < x < 1.$$

By calculating the principal value of the definite integral, we will obtain

$$\int_{-1}^{1}\frac{d\xi}{\xi-x}=\lim_{\varepsilon\to 0}\left(\int_{-1}^{x-\varepsilon}\frac{d\xi}{\xi-x}+\int_{x+\varepsilon}^{1}\frac{d\xi}{\xi-x}\right)=$$

$$=\lim_{\varepsilon\to 0}\left[\ln(x-\xi)\Big|_{-1}^{x-\varepsilon}+\ln(\xi-x)\Big|_{x+\varepsilon}^{1}\right]=\lim_{\varepsilon\to 0}\ln\frac{\varepsilon(1-x)}{(1+x)\varepsilon}=\ln\frac{1-x}{1+x}. \qquad (348)$$

Further,

$$\int_{-1}^{1}\frac{d\xi}{\xi-\frac{1}{x}}=\ln\left(\frac{1}{x}-\xi\right)\Big|_{-1}^{1}=\ln\frac{1-x}{1+x}. \qquad (349)$$

Substituting (348) and (349) into (347), we will find

$$J_1=0. \qquad (350)$$

Substituting (350), (346), and (345) into (344), we will find

$$p(\pi+\theta)=\frac{\sqrt{2}}{2\pi\sqrt{\cos\theta-\cos\alpha}}\times$$

$$\times\left\{\frac{A\sin^2\frac{\alpha}{2}\,\mathrm{tg}\,\frac{\alpha}{2}}{\cos\frac{\theta}{2}}\left[2c_2+\left(1+\cos^2\frac{\alpha}{2}\right)c_4\right]+P\cos\frac{\theta}{2}\right\}. \qquad (351)$$

Excluding from equations (361), c_1, c_3 and c_5, we will obtain equations

$$\left.\begin{aligned}c_2\sec^2\frac{\theta}{2}+c_4\,\mathrm{tg}^2\,\frac{\theta}{2}&=-\mathrm{ctg}^2\,\frac{\alpha}{2},\\ -c_2\sec^2\frac{\theta}{2}+c_4&=\mathrm{ctg}\,\frac{\alpha}{2}+2\,\mathrm{ctg}^2\,\frac{\alpha}{2},\end{aligned}\right\}$$

whence

$$c_2=-\mathrm{ctg}^2\,\frac{\alpha}{2}\cos^2\frac{\theta}{2}\left(1+\sec^2\frac{\alpha}{2}\sin^2\frac{\theta}{2}\right), \qquad (352)$$

$$c_4=\mathrm{ctg}^2\,\frac{\alpha}{2}\sec^2\frac{\alpha}{2}\cos^2\frac{\theta}{2}.$$

Substituting (352) into (351), we will obtain the unknown solution

$$p(\pi+\theta)=\frac{\sqrt{2}\cos\frac{\theta}{2}}{2\pi\sqrt{\cos\theta-\cos\alpha}}\left[A\left(\cos\theta-\cos^2\frac{\alpha}{2}\right)+P\right], \qquad (353)$$

$$-\alpha<\theta<\alpha.$$

As we see, when ϑ approaches $-\alpha$ or to $+\alpha$, r approaches infinity. Only in the special case when

$$P = A\sin^2\frac{\alpha}{2}, \tag{354}$$

do we obtain according to (353) the solution

$$p(\pi+\vartheta) = \frac{\sqrt{2}A}{2\alpha}\cos\frac{\vartheta}{2}\sqrt{\cos\vartheta-\cos\alpha}, \tag{355}$$

limited in the whole interval $-\alpha<\vartheta<\alpha$.

§ 6. Equation of the Contact Problem in the Presence of Friction Between Compressible Bodies

Let us now examine the equation

$$\int_0^x p(t)\,dt + \nu\int_{-a}^{a} p(t)\ln\frac{1}{|t-x|}\,dt = f(x), \quad |x|<a, \tag{356}$$

where ν — certain constant.

Just as earlier, let us introduce into the consideration the logarithmic potential of the simple layer.

$$V(x,y) = \int_{-a}^{a} p(t)\ln\frac{1}{r}\,dt, \quad r = \sqrt{(t-x)^2+y^2}. \tag{357}$$

As we already know, function $V(x, y)$ satisfies relations

$$\left(\frac{\partial V}{\partial y}\right)_{y=+0} = -\pi p(x) \text{ when } |x|<a, \quad \left(\frac{\partial V}{\partial y}\right)_{y=+0} = 0 \text{ when } |x|>a. \tag{358}$$

Equation (356), according to (357) can be given the form

$$\int_0^x p(t)\,dt + \nu V(x,0) = f(x), \quad |x|<a,$$

or, if one were to differentiate this relation with respect to x,

$$p(x)+\nu\left(\frac{\partial V}{\partial x}\right)_{y=0}=f'(x), \quad |x|<a. \tag{359}$$

Taking into account (358), we can give to condition (359) the form

$$\frac{\partial V}{\partial y}-\pi\nu\frac{\partial V}{\partial x}=-\pi f'(x) \quad \text{when } |x|<a,\ y=+0. \tag{360}$$

Let us examine in the upper half-plane $y > 0$ the function of the complex variable z:

$$F(z)=\int_{-a}^{a}\frac{p(t)\,dt}{t-z}=\frac{\partial V}{\partial x}-i\frac{\partial V}{\partial y}. \tag{361}$$

According to (358) and (360) function $F(z)$ will satisfy boundary conditions

$$\left.\begin{array}{ll}\pi\nu\,\mathrm{Re}\,F(z)+\mathrm{Im}\,F(z)=\pi f'(x) & \text{when } |x|<a,\ y=+0,\\ \mathrm{Im}\,F(z)=0 & \text{when } |x|>a,\ y=+0.\end{array}\right\} \tag{362}$$

In the neighborhood of the point at infinity function $F(z)$ will have the expansion

$$F(z)=-\frac{P}{z}+\ldots, \quad P=\int_{-a}^{a}p(t)\,dt. \tag{363}$$

By knowing function $F(z)$, we will find the solution of the initial equation (356) $p(x)$ by formula

$$p(x)=\frac{1}{\pi}\left[\mathrm{Im}\,F(z)\right]_{y=+0}, \quad |x|<a, \tag{364}$$

as follows from (358) and (361).

Thus, the solution of equation (356) is reduced to the construction of the function of the complex variable $F(z)$ in the upper half-plane $y > 0$ according to boundary conditions (36) [sic] and condition (363).

We will look for function $F(z)$ in the form

$$F(z)=(z+a)^{-\frac{1}{2}-\gamma}(z-a)^{-\frac{1}{2}+\gamma}\Phi(z), \quad -\frac{1}{2}<\gamma<\frac{1}{2}, \tag{365}$$

thus, reducing the detecting of function $F(z)$ to the detecting of function $\Phi(z)$. Let us deduce the condition which should be satisfied by function $\Phi(z)$. The difference $z - a$ can be presented in the form

$$z-a=\rho e^{i\vartheta}, \tag{366}$$

where

$$\rho=\sqrt{(x-a)^2+y^2}, \quad \vartheta=\operatorname{arc\,tg}\frac{y}{x-a} \tag{367}$$

(Figure 4). In this case we consider that ϑ changes within limits of 0 to π when point z is in the upper half-plane. From (366) we find

$$\begin{aligned}(z-a)^{-\frac{1}{2}+\gamma}&=\rho^{-\frac{1}{2}+\gamma}e^{i\left(-\frac{1}{2}+\gamma\right)\vartheta}=\\ &=\rho^{-\frac{1}{2}+\gamma}\left[\cos\left(\frac{1}{2}-\gamma\right)\vartheta-\right.\\ &\left.-i\sin\left(\frac{1}{2}-\gamma\right)\vartheta\right].\end{aligned} \tag{368}$$

As can be seen from relation (367),

$$\begin{gathered}\rho=|x-a| \text{ when } y=+0, \quad \vartheta=0 \text{ when } x>a,\ y=+0,\\ \vartheta=\pi \text{ when } x<a,\ y=+0.\end{gathered} \tag{369}$$

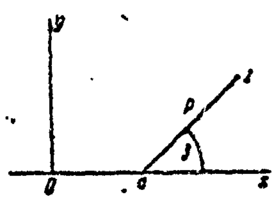

Fig. 4.

Assuming in (368) $y = +0$, let us find on the basis of (369)

$$\left.\begin{aligned}&(z-a)^{-\frac{1}{2}+\gamma}=|x-a|^{-\frac{1}{2}+\gamma} \text{ when } x>a,\ y=+0,\\ &(z-a)^{-\frac{1}{2}+\gamma}=|x-a|^{-\frac{1}{2}+\gamma}(\sin\pi\gamma-i\cos\pi\gamma)\\ &\qquad\qquad \text{when } x<a,\ y=+0.\end{aligned}\right\} \tag{370}$$

Changing in (370) the sign at a and γ, we will find

$$\left.\begin{aligned}&(z+a)^{-\frac{1}{2}-\gamma}=|x+a|^{-\frac{1}{2}-\gamma} \quad \text{when } x>-a,\ y=+0,\\&(z+a)^{-\frac{1}{2}-\gamma}=|x+a|^{-\frac{1}{2}-\gamma}(-\sin\pi\gamma-i\cos\pi\gamma)\\&\qquad\qquad \text{when } x<-a,\ y=+0.\end{aligned}\right\} \tag{371}$$

From (370) and (371) we find

$$\left.\begin{aligned}&(z+a)^{-\frac{1}{2}-\gamma}(z-a)^{-\frac{1}{2}+\gamma}=(x+a)^{-\frac{1}{2}-\gamma}(x-a)^{-\frac{1}{2}+\gamma}\\&\qquad\qquad \text{when } x>a,\ y=+0,\\&(z+a)^{-\frac{1}{2}-\gamma}(z-a)^{-\frac{1}{2}+\gamma}=\\&\qquad =(a+x)^{-\frac{1}{2}-\gamma}(a-x)^{-\frac{1}{2}+\gamma}(\sin\pi\gamma-i\cos\pi\gamma)\\&\qquad\qquad \text{when } -a<x<a,\ y=+0,\\&(z+a)^{-\frac{1}{2}-\gamma}(z-a)^{-\frac{1}{2}+\gamma}=-(-a-x)^{-\frac{1}{2}-\gamma}(a-x)^{-\frac{1}{2}+\gamma}\\&\qquad\qquad \text{when } x<-a,\ y=+0.\end{aligned}\right\} \tag{372}$$

Assuming in (365) $y = +0$, we will find according to (372)

$$\left.\begin{aligned}&\operatorname{Re}F(z)+i\operatorname{Im}F(z)=\\&\qquad =(x+a)^{-\frac{1}{2}-\gamma}(x-a)^{-\frac{1}{2}+\gamma}[\operatorname{Re}\Phi(z)+i\operatorname{Im}\Phi(z)]\\&\qquad\qquad \text{when } x>a,\ y=+0,\\&\operatorname{Re}F(z)+i\operatorname{Im}F(z)=\\&\qquad =(a+x)^{-\frac{1}{2}-\gamma}(a-x)^{-\frac{1}{2}+\gamma}(\sin\pi\gamma-i\cos\pi\gamma)\times\\&\times[\operatorname{Re}\Phi(z)+i\operatorname{Im}\Phi(z)] \quad \text{when } -a<x<a,\ y=+0,\\&\operatorname{Re}F(z)+i\operatorname{Im}F(z)=\\&\qquad =-(-a-x)^{-\frac{1}{2}-\gamma}(a-x)^{-\frac{1}{2}+\gamma}[\operatorname{Re}\Phi(z)+i\operatorname{Im}\Phi(z)]\\&\qquad\qquad \text{when } x<-a,\ y=+0.\end{aligned}\right\} \tag{373}$$

The first and third of relations (373) directly give:

$$\left.\begin{aligned}&\operatorname{Im}\Phi(z)=(x+a)^{\frac{1}{2}+\gamma}(x-a)^{\frac{1}{2}-\gamma}\operatorname{Im}F(z)\\&\qquad\qquad \text{when } x>a,\ y=+0,\\&\operatorname{Im}\Phi(z)=-(-a-x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma}\operatorname{Im}F(z)\\&\qquad\qquad \text{when } x<-a,\ y=+0.\end{aligned}\right\} \tag{374}$$

Multiplying both sides of the second of relations (373) by

$\sin \pi\gamma + i \cos \pi\gamma$, we will find

$$\operatorname{Im} \Phi(z) = (a+x)^{\frac{1}{2}+\gamma} (a-x)^{\frac{1}{2}-\gamma} [\cos \pi\gamma \operatorname{Re} F(z) + \sin \pi\gamma \operatorname{Im} F(z)] \quad \text{when } -a < x < a,\ y = +0. \tag{375}$$

The constant γ up till now has remained indefinite for us. Let us now put

$$\operatorname{tg} \pi\gamma = \frac{1}{\pi\nu}, \quad -\frac{1}{2} < \gamma < \frac{1}{2}. \tag{376}$$

Using relation (376), it is possible to give to condition (375) the form

$$\operatorname{Im} \Phi(z) = \frac{\cos \pi\gamma}{\pi\nu} (a+x)^{\frac{1}{2}+\gamma} (a-x)^{\frac{1}{2}-\gamma} [\pi\nu \operatorname{Re} F(z) + \operatorname{Im} F(z)] \quad \text{when } -a < x < a,\ y = +0. \tag{377}$$

Substituting (362) into (374) and (377), we will obtain for function $\Phi(z)$ the boundary conditions

$$\left.\begin{aligned} &\operatorname{Im} \Phi(z) = \frac{\cos \pi\gamma}{\nu} (a+x)^{\frac{1}{2}+\gamma} (a-x)^{\frac{1}{2}-\gamma} f'(x) && \text{when } |x| < a,\ y = +0, \\ &\operatorname{Im} \Phi(z) = 0 && \text{when } |x| > a,\ y = +0. \end{aligned}\right\} \tag{378}$$

Let us clarify further the behavior of function $\Phi(z)$ in the neighborhood of the point at infinity. From (365) we find

$$\Phi(z) = (z+a)^{\frac{1}{2}+\gamma} (z-a)^{\frac{1}{2}-\gamma} F(z) = z\left(1+\frac{a}{z}\right)^{\frac{1}{2}+\gamma}\left(1-\frac{a}{z}\right)^{\frac{1}{2}-\gamma} F(z). \tag{379}$$

But, according to (363), we should have

$$[zF(z)]_{z=\infty} = -P. \tag{380}$$

From (379) and (380) it follows that

$$\Phi(\infty) = -P. \tag{381}$$

We will look for function $\Phi(z)$ in the form

$$\Phi(z) = \int_{-a}^{a} \frac{q(t)\,dt}{t-z} + c, \tag{382}$$

where c — real constant.

As we already know,

$$\begin{aligned} \operatorname{Im}\Phi(z) &= \pi q(x) \quad \text{when } |x| < a,\ y = +0, \\ \operatorname{Im}\Phi(z) &= 0 \qquad\ \text{when } |x| > a,\ y = +0 \end{aligned} \tag{383}$$

(see, for example, formulas (361) and (358)).

By comparing (383) and (378), we will find that boundary conditions (378) will be satisfied if one were to assume that

$$q(x) = \frac{\cos\pi\gamma}{\pi\nu}(a+x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma} f'(x). \tag{384}$$

Further, as can be seen from (381) and (382), condition (381) will be carried out if one were to assume that

$$c = -P. \tag{385}$$

Substituting (384) and (385) into (382), we will find

$$\Phi(z) = \frac{\cos\pi\gamma}{\pi\nu} \int_{-a}^{a} \frac{(a+t)^{\frac{1}{2}+\gamma}(a-t)^{\frac{1}{2}-\gamma} f'(t)\,dt}{t-z} - P. \tag{386}$$

The second of relations (373) gives

$$\operatorname{Im} F(z) = (a+x)^{-\frac{1}{2}-\gamma}(a-x)^{-\frac{1}{2}+\gamma}[\sin\pi\gamma \operatorname{Im}\Phi(z) - \cos\pi\gamma \operatorname{Re}\Phi(z)] \\ \text{when } |x| < a,\ y = +0. \tag{387}$$

From (364) and (387) we find

$$p(x) = \frac{\sin\pi\gamma\,[\operatorname{Im}\Phi(z)]_{y=+0} - \cos\pi\gamma\,[\operatorname{Re}\Phi(z)]_{y=+0}}{\pi(a+x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma}} \quad \text{when } |x| < a. \tag{388}$$

Substituting (384) into (383), we will find

$$[\operatorname{Im} \Phi(z)]_{y\to+0} = \frac{\cos\pi\gamma}{\nu}(a+x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma} f'(x) \text{ when } |x|<a. \tag{389}$$

From (386) it follows directly that

$$[\operatorname{Re} \Phi(z)]_{y\to+0} = \frac{\cos\pi\gamma}{\pi\nu}\int_{-a}^{a}\frac{(a+t)^{\frac{1}{2}+\gamma}(a-t)^{\frac{1}{2}-\gamma} f'(t)\,dt}{t-x} - P \text{ when } |x|<a. \tag{390}$$

Substituting (389) and (390) into (388), we will find the unknown solution of equation (356):

$$p(x) = \frac{\sin\pi\gamma\cos\pi\gamma}{\pi\nu} f'(x) - \frac{\dfrac{\cos^2\pi\gamma}{\pi\nu}\displaystyle\int_{-a}^{a}\frac{(a+t)^{\frac{1}{2}+\gamma}(a-t)^{\frac{1}{2}-\gamma} f'(t)\,dt}{t-x} - P\cos\pi\gamma}{\pi(a+x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma}}, \ |x|<a, \tag{391}$$

where

$$P = \int_{-a}^{a} p(t)\,dt, \quad \gamma = \frac{1}{\pi}\operatorname{arc\,tg}\frac{1}{\pi\nu}\left(-\frac{1}{2}<\gamma<\frac{1}{2}\right) \tag{392}$$

according to (363) and (376).

§ 7. Equation of the Problem About the Compression of Elastic Bodies Bounded by Cylindrical Surfaces

Let us now examine the equation[1] to which the problem about the compression of two elastic bodies bounded by circular cylindrical surfaces leads:

$$\lambda(x)g(x) + \int_{-a}^{a}\frac{g'(t)\,dt}{t-x} = f(x), \qquad -a<x<a, \tag{393}$$

[1]For a numberical solution of equation (393) the method of finite differences is very convenient - see Appendix II.

where $\lambda(x)$ and $f(x)$ are assinged functions (we will assume below that $\lambda(x) \neq 0$ when $-a < x < a$, $f(-x) = f(x)$), and $g(x)$— is the function subject to determination. Giving rise to equation (411) is also the theory of the airfoil of finite span, developed by Prandtl, in consequence of which this equation is called the Prandtl equation. We give below the solution of this equation for the case when function $\lambda(x)$ has the form

$$\lambda(x) = \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_n x^n} \sqrt{a^2 - x^2}, \tag{394}$$

where both polynomials entering into (394) do not have real roots in the interval $-a < x < a$.

In general for the assigned function $\lambda(x)$ it is possible to construct a quite similar approximate expression of the form (394), having taken the number n sufficiently large.

We adhere below to the method of the solution of equation (393) proposed I. Vekua[1]. By means of this method the solution of equation (41) is reduced to the integration of the differential second-order equation.

In § 2 we showed that equation

$$\int_{-a}^{a} p(t) \ln \frac{1}{|t - x|}\, dt = f(x), \quad -a < x < a, \tag{395}$$

has the solution

$$p(x) = \frac{1}{\pi \sqrt{a^2 - x^2}} \left[P - \frac{1}{\pi} \int_{-a}^{a} \frac{f'(t) \sqrt{a^2 - t^2}}{t - x}\, dt \right]$$

(see formula (115)), or, if one were to use identiy

[1]See I. N. Vekua, on the integro-differential Prandtl equation. Applied mathematics and mechanics, Vol. 9, No. 2, 1945.

$$\frac{\sqrt{a^2-t^2}}{t-x}=\frac{a^2-x^2}{\sqrt{a^2-t^2}(t-x)}-\frac{t+x}{\sqrt{a^2-t^2}},$$

$$p(x)=\frac{1}{\pi\sqrt{a^2-x^2}}\left[P+\frac{1}{\pi}\int_{-a}^{a}\frac{f'(t)\,t\,dt}{\sqrt{a^2-t^2}}-\right.$$

$$\left.-\frac{1}{\pi}(a^2-x^2)\int_{-a}^{a}\frac{f'(t)dt}{\sqrt{a^2-t^2}(t-x)}\right].$$

Since

$$\int_{-a}^{a}\frac{f'(t)dt}{\sqrt{a^2-t^2}}=0,$$

if $f(x)$ is the even function.

This solution will be limited in points $x = -a$ and $x = a$ then and only when

$$P=-\frac{1}{\pi}\int_{-a}^{a}\frac{f'(t)\,t\,dt}{\sqrt{a^2-t^2}}.$$

In this case the solution of equation (395) takes the form

$$p(x)=-\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{f'(t)dt}{\sqrt{a^2-t^2}(t-x)},\qquad -a<x<a. \tag{396}$$

The same solution, limited in points $x = -a$ and $x = a$, will be had by the equation which we will obtain by differentiating with respect to x both sides of equation (395). When $t > x$ we find

$$\frac{d}{dx}\ln\frac{1}{|t-x|}=\frac{d}{dx}\ln\frac{1}{t-x}=\frac{1}{t-x},$$

when $t < x$:

$$\frac{d}{dx}\ln\frac{1}{|t-x|}=\frac{d}{dx}\ln\frac{1}{x-t}=\frac{1}{t-x}.$$

Thus,

$$\frac{d}{dx}\ln\frac{1}{|t-x|}=\frac{1}{t-x}.$$

Differentiating with respect to x both sides of equation (395), we will obtain the equation

$$\int_{-a}^{a}\frac{p(t)dt}{t-x}=f'(x),\qquad -a<x<a, \tag{397}$$

which has the solution (396).

Let us present now equation (393) in the form

$$\int_{-a}^{a} \frac{g'(t)\,dt}{t-x} = f(x) - \lambda(x) g(x), \qquad -a < x < a \tag{398}$$

we will temporarily examine the right side of this equation as the known function. Then, according to formula (396) for the solution of equation (397), we will be able to find from (398) derivative $g'(x)$, which stands under the integral sign:

$$g'(x) = -\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a} \frac{f(t)-\lambda(t)g(t)}{\sqrt{a^2-t^2}\,(t-x)}\,dt, \qquad -a < x < a. \tag{399}$$

Substituting (394), into (399), we will find

$$g'(x) = \frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a} \frac{g(t)(a_0+a_1t+\ldots+a_nt^n)}{(t-x)(b_0+b_1t+\ldots+b_nt^n)}\,dt + F(x), \tag{400}$$

$$-a < x < a,$$

where

$$F(x) = -\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a} \frac{f(t)\,dt}{\sqrt{a^2-t^2}\,(t-x)}. \tag{401}$$

Relation (400) can be presented in form

$$\pi^2\frac{g'(x)-F(x)}{\sqrt{a^2-x^2}} = \frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_nx^n}\int_{-a}^{a}\frac{g(t)\,dt}{t-x} + \int_{-a}^{a} g(t) R(t,x)\,dt, \tag{402}$$

where

$$R(t,x) = \frac{1}{t-x}\left[\frac{a_0+a_1t+\ldots+a_nt^n}{b_0+b_1t+\ldots+b_nt^n} - \frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_nx^n}\right], \tag{403}$$

or

$$R(t,x) = \frac{P_0(t)+P_1(t)x+\ldots+P_{n-1}(t)x^{n-1}}{(b_0+b_1t+\ldots+b_nt^n)(b_0+b_1x+\ldots+b_nx^n)}, \tag{404}$$

where $P_0(t), P_1(t), \ldots, P_{n-1}(t)$ — polynomials in t, the coefficients of which are easily calculated. Actually, by comparing (403) and (404), we will find

$$(t-x)[P_0(t)+P_1(t)x+\ldots+P_{n-1}(t)x^{n-1}]=$$
$$=(a_0+a_1t+\ldots+a_nt^n)(b_0+b_1x+\ldots+b_nx^n)-$$
$$-(a_0+a_1x+\ldots+a_nx^n)(b_0+b_1t+\ldots+b_nt^n). \qquad (405)$$

By comparing components containing identical powers of x in the left and right sides of relation (405), we will obtain the equations

$$\left.\begin{aligned}
tP_0(t)&=b_0(a_0+a_1t+\ldots+a_nt^n)-\\
&\quad-a_0(b_0+b_1t+\ldots+b_nt^n),\\
tP_1(t)-P_0(t)&=b_1(a_0+a_1t+\ldots+a_nt^n)-\\
&\quad-a_1(b_0+b_1t+\ldots+b_nt^n),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
tP_{n-1}(t)-P_{n-2}(t)&=b_{n-1}(a_0+a_1t+\ldots+a_nt^n)-\\
&\quad-a_{n-1}(b_0+b_1t+\ldots+b_nt^n),\\
-P_{n-1}(t)&=b_n(a_0+a_1t+\ldots+a_nt^n)-\\
&\quad-a_n(b_0+b_1t+\ldots+b_nt^n),
\end{aligned}\right\} \qquad (406)$$

which determine polynomials $P_0(t), P_1(t), \ldots, P_{n-1}(t)$. From the first $n-1$ equations (406) we will successively find

$$\left.\begin{aligned}
P_0(t)&=b_0(a_1+a_2t+\ldots+a_nt^{n-1})-\\
&\quad-a_0(b_1+b_2t+\ldots+b_nt^{n-1}),\\
P_1(t)&=(b_0+b_1t)(a_2+a_3t+\ldots+a_nt^{n-2})-\\
&\quad-(a_0+a_1t)(b_2+b_3t+\ldots+b_nt^{n-2}),\\
P_2(t)&=(b_0+b_1t+b_2t^2)(a_3+a_4t+\ldots+a_nt^{n-3})-\\
&\quad-(a_0+a_1t+a_2t^2)(b_3+b_4t+\ldots+b_nt^{n-3}),\\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\
P_{n-1}(t)&=(b_0+b_1t+\ldots+b_{n-1}t^{n-1})a_n-\\
&\quad-(a_0+a_1t+\ldots+a_{n-1}t^{n-1})b_n,
\end{aligned}\right\} \qquad (407)$$

and the last of equations (406) is a corollary of the others. Substituting (404) into (402), we will find

$$\pi^2\frac{g'(x)-F(x)}{\sqrt{a^2-x^2}}=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_nx^n}\int_{-a}^{a}\frac{g(t)\,dt}{t-x}+\frac{\alpha_0+\alpha_1x+\ldots+\alpha_{n-1}x^{n-1}}{b_0+b_1x+\ldots+b_nx^n}, \qquad (408)$$

where

$$\alpha_k=\int_{-a}^{a}\frac{g(t)P_k(t)\,dt}{b_0+b_1t+\ldots+b_nt^n}, \qquad k=0,1,\ldots,n-1. \qquad (409)$$

Relation (409) connect the unknown as yet constants a_0, $a_1, \ldots, a_{n-1}$ with the unknown function $g(x)$.

Let us represent (408) in the form

$$\pi^2 \frac{(b_0 + b_1 x + \ldots + b_n x^n)[g'(x) - F(x)]}{(a_0 + a_1 x + \ldots + a_n x^n)\sqrt{a^2 - x^2}} = \int_{-a}^{a} \frac{g(t)\,dt}{t - x} + \frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{n-1} x^{n-1}}{a_0 + a_1 x + \ldots + a_n x^n},$$

or, according to (394),

$$\pi^2 \frac{g'(x) - F(x)}{\lambda(x)} = \int_{-a}^{a} \frac{g(t)\,dt}{t - x} + R(x), \tag{410}$$

where $R(x)$ – rational function:

$$R(x) = \frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{n-1} x^{n-1}}{a_0 + a_1 x + \ldots + a_n x^n}. \tag{411}$$

Integrating by parts, we will find

$$\int_{-a}^{a} \frac{g(t)\,dt}{t - x} = -\int_{-a}^{a} g(t)\,d\left(\ln \frac{1}{|t - x|}\right) = \int_{-a}^{a} g'(t) \ln \frac{1}{|t - x|}\,dt + g(a)\ln(a - x) - g(-a)\ln(a + x). \tag{412}$$

Differentiating both sides of the obtained relationship with respect to x, we will have

$$\frac{d}{dx}\int_{-a}^{a} \frac{g(t)\,dt}{t - x} = \int_{-a}^{a} \frac{g'(t)\,dt}{t - x} - \frac{g(a)}{a - x} - \frac{g(-a)}{a + x}. \tag{413}$$

Substituting (398) into (413), we will obtain

$$\frac{d}{dx}\int_{-a}^{a} \frac{g(t)\,dt}{t - x} = f(x) - \lambda(x) g(x) - \frac{g(a)}{a - x} - \frac{g(-a)}{a + x}. \tag{414}$$

Differentiating with respect to x both sides of relation (410) and

taking into account (414), we will find

$$\pi^2 \frac{d}{dx}\left[\frac{g'(x)-F(x)}{\lambda(x)}\right]=f(x)-\lambda(x)g(x)+R'(x)-\frac{g(a)}{a-x}-\frac{g(-a)}{a+x}. \qquad (415)$$

Thus, the solution of the initial equation (393) is reduced by us to integration of the differential equation (415). This differential equation is integrated in quadratures. Actually, assuming

$$\mu(x)=\frac{1}{\pi}\int_0^x \lambda(t)\,dt, \qquad (416)$$

we will have

$$\lambda(x)=\pi\frac{d\mu}{dx}, \quad \pi\frac{g'(x)}{\lambda(x)}=\frac{dg}{dx}\frac{dx}{d\mu}=\frac{dg}{d\mu},$$

$$\frac{\pi^2}{\lambda(x)}\frac{d}{dx}\left[\frac{g'(x)}{\lambda(x)}\right]=\frac{d}{dx}\left(\frac{dg}{d\mu}\right)\frac{dx}{d\mu}=\frac{d^2g}{d\mu^2}.$$

Thus, dividing both sides of equation (415) by $\lambda(x)$, we will obtain the equation

$$\frac{d^2g}{d\mu^2}+g=\frac{1}{\lambda(x)}\left\{\pi^2\frac{d}{dx}\left[\frac{F(x)}{\lambda(x)}\right]+f(x)+R'(x)-\frac{g(a)}{a-x}-\frac{g(-a)}{a+x}\right\},$$

or

$$\frac{d^2g}{d\mu^2}+g=\Phi(x), \qquad (417)$$

where

$$\Phi(x)=\frac{1}{\lambda(x)}\left\{\pi^2\frac{d}{dx}\left[\frac{F(x)}{\lambda(x)}\right]+f(x)+R'(x)-\frac{g(a)}{a-x}-\frac{g(-a)}{a+x}\right\}. \qquad (418)$$

Following the method of variation of arbitrary constants, we will look for the solution of equation (417) in the form

$$g=f_1(\mu)\cos\mu+f_2(\mu)\sin\mu, \qquad (419)$$

having subordinated unknown functions $f_1(\mu)$ and $f_2(\mu)$ to condition

$$f_1'(\mu)\cos\mu+f_2'(\mu)\sin\mu=0. \qquad (420)$$

Differentiating (419) and taking into account (420), we will find

$$\left.\begin{aligned}\frac{dg}{d\mu}&=-f_1(\mu)\sin\mu+f_2(\mu)\cos\mu,\\ \frac{d^2g}{d\mu^2}&=-f_1'(\mu)\sin\mu+f_2'(\mu)\cos\mu-f_1(\mu)\cos\mu-f_2(\mu)\sin\mu.\end{aligned}\right\} \tag{421}$$

Substituting (419) and (421) into (417), we will find

$$-f_1'(\mu)\sin\mu+f_2'(\mu)\cos\mu=\Phi(x). \tag{422}$$

From (420) and (422) we will find

$$f_1'(\mu)=-\Phi(x)\sin\mu,\quad f_2'(\mu)=\Phi(x)\cos\mu. \tag{423}$$

Hence

$$\begin{aligned}f_1(\mu)&=-\int_0^\mu\Phi(t)\sin\mu^*\,d\mu^*+c_1,\\ f_2(\mu)&=\int_0^\mu\Phi(t)\cos\mu^*\,d\mu^*+c_2,\end{aligned} \tag{424}$$

where variables t and μ^* according to (416) are connected by relation

$$\mu^*(t)=\frac{1}{\varkappa}\int_0^t\lambda(t)\,dt, \tag{425}$$

and c_1 and c_2 are arbitrary constants.

Putting (424) in (419), we will find:

$$g=\int_0^\mu\Phi(t)(\cos\mu^*\sin\mu-\sin\mu^*\cos\mu)\,d\mu^*+c_1\cos\mu+c_2\sin\mu,$$

or

$$g=\int_0^\mu\Phi(t)\sin(\mu-\mu^*)\,d\mu^*+c_1\cos\mu+c_2\sin\mu. \tag{426}$$

Passing in (426) from variable μ^* to variable t, we will find in accordance with relations (416) and (425)

$$g(x)=\frac{1}{\varkappa}\int_0^x\Phi(t)\sin[\mu(x)-\mu(t)]\lambda(t)\,dt+c_1\cos\mu(x)+c_2\sin\mu(x). \tag{427}$$

Substituting (436) into (445), we will obtain

$$g(x)=\frac{1}{\pi}\int_0^x \frac{d}{dt}\left[\pi^2\frac{F(t)}{\lambda(t)}+R(t)\right]\sin[\mu(x)-\mu(t)]\,dt+$$
$$+\frac{1}{\pi}\int_0^x\left[f(t)-\frac{g(a)}{a-t}-\frac{g(-a)}{a+t}\right]\sin[\mu(x)-\mu(t)]\,dt+$$
$$+c_1\cos\mu(x)+c_2\sin\mu(x). \qquad (428)$$

Integrating by parts and taking into account (416), we will find

$$\int_0^x\frac{d}{dt}\left[\pi^2\frac{F(t)}{\lambda(t)}R(t)\right]\sin[\mu(x)-\mu(t)]\,dt=$$
$$=-\int_0^x\left[\pi^2\frac{F(t)}{\lambda(t)}+R(t)\right]\cos[\mu(x)-\mu(t)]\mu'(t)\,dt-$$
$$-\left[\pi^2\frac{F(0)}{\lambda(0)}+R(0)\right]\sin\mu(x)=-\frac{1}{\pi}\int_0^x[\pi^2F(t)+\lambda(t)R(t)]\times$$
$$\times\cos[\mu(x)-\mu(t)]\,dt-\left[\pi^2\frac{F(0)}{\lambda(0)}+R(0)\right]\sin\mu(x). \qquad (429)$$

Substituting (447) into (446) and including constant $-\pi\frac{F(0)}{\lambda(0)}-\frac{1}{\pi}R(0)$ into the arbitrary constant c_2, we will obtain the following final expression for the unknown function $g(x)$:

$$g(x)=-\frac{1}{\pi^2}\int_0^x[\pi^2F(t)+\lambda(t)R(t)]\cos[\mu(x)-\mu(t)]\,dt+$$
$$+\frac{1}{\pi}\int_0^x\left[f(t)-\frac{g(a)}{a-t}-\frac{g(-a)}{a+t}\right]\sin[\mu(x)-\mu(t)]\,dt+$$
$$+c_1\cos\mu(x)+c_2\sin\mu(x), \qquad (430)$$

where functions $F(x)$, $R(x)$ and $\mu(x)$ are determined by formulas (401) (411) and (416).

Assuming in (430) $x = a$ and $x = -a$, we will obtain the equations for the determination of constants c_1 and c_2 appearing in formula (430):

$$\left.\begin{aligned}&c_1\cos\mu(a)+c_2\sin\mu(a)=g(a)+\frac{1}{\pi^2}\int_0^a[\pi^2F(t)+\lambda(t)R(t)]\times\\&\quad\times\cos[\mu(a)-\mu(t)]\,dt-\frac{1}{\pi}\int_0^a\left[f(t)-\frac{g(a)}{a-t}-\frac{g(-a)}{a+t}\right]\times\\&\qquad\times\sin[\mu(a)-\mu(t)]\,dt,\\&c_1\cos\mu(-a)+c_2\sin\mu(-a)=g(-a)+\frac{1}{\pi^2}\times\end{aligned}\right\} \qquad (431)$$

$$\left.\begin{aligned} &\times \int_{0}^{-a} [\pi^2 F(t) + \lambda(t) R(t)] \cos[\mu(-a) - \mu(t)]\, dt - \\ &- \frac{1}{\pi} \int_{0}^{-a} \left[f(t) - \frac{g(a)}{a-t} - \frac{g(-a)}{a+t} \right] \sin[\mu(-a) - \mu(t)]\, dt. \end{aligned}\right\} \quad (431 \text{ cont'd})$$

Substituting (430) into (409), we will obtain the system of n linear equations for the determination of constants $\alpha_0, \alpha_1, \ldots, \alpha_{n-1}$ which appear in the expression for $R(x)$.

Boundary values of the unknown function $g(a)$ and $g(-a)$ appearing in formula (430) are determined by supplementary conditions, which result from the formulation of a certain problem leading to equation (393).

The same procedure by which we obtained the solution of equation (393) can be used to obtain the solution of equation

$$\lambda(x) g'(x) + \int_{-a}^{a} \frac{g(t)\, dt}{t-x} = f(x), \quad -a < x < a, \tag{432}$$

if in this equation $\lambda(x)$ has the form (394).

Having presented equation (432) in the form

$$\int_{-a}^{a} \frac{g(t)\, dt}{t-x} = f(x) - \lambda(x) g'(x), \quad -a < x < a \tag{433}$$

and temporarily examining the right side of this equation as the known function, we will find

$$g(x) = -\frac{1}{\pi^2} \sqrt{a^2 - x^2} \int_{-a}^{a} \frac{f(t) - \lambda(t) g'(t)}{\sqrt{a^2 - t^2}\,(t-x)}\, dt, \quad -a < x < a, \tag{434}$$

in accordance with formula (396) for the solution of equation (397).

Substituting (394) into (434), we will find

$$g(x)=\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{g'(t)(a_0+a_1t+\ldots+a_nt^n)}{(t-x)(b_0+b_1t+\ldots+b_nt^n)}\,dt+F(x), \tag{435}$$
$$-a<x<a,$$

where

$$F(x)=-\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{f(t)\,dt}{\sqrt{a^2-t^2}\,(t-x)}. \tag{436}$$

Relation (435) can be presented in the form

$$\pi^2\frac{g(x)-F(x)}{\sqrt{a^2-x^2}}=$$
$$=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_nx^n}\int_{-a}^{a}\frac{g'(t)\,dt}{t-x}+\int_{-a}^{a}g'(t)R(t,x)\,dt, \tag{437}$$

where $R(t, x)$ is the function determined by formula (403) or its equivalent formula (404). Substituting (404) into (437), we will find

$$\pi^2\frac{g(x)-F(x)}{\sqrt{a^2-x^2}}=$$
$$=\frac{a_0+a_1x+\ldots+a_nx^n}{b_0+b_1x+\ldots+b_nx^n}\int_{-a}^{a}\frac{g'(t)\,dt}{t-x}+\frac{\beta_0+\beta_1x+\ldots+\beta_{n-1}x^{n-1}}{b_0+b_1x+\ldots+b_nx^n}, \tag{438}$$

where

$$\beta_k=\int_{-a}^{a}\frac{g'(t)P_k(t)\,dt}{b_0+b_1t+\ldots+b_nt^n}, \tag{439}$$
$$k=0,1,\ldots,n-1,$$

and $P_0(t), P_1(t), \ldots, P_{n-1}(t)$ are polynomials determined by formulas (407).

Differentiating both sides of equation (433) with respect to x, we will find according to formula (413)

$$\int_{-a}^{a}\frac{g'(t)\,dt}{t-x}=f'(x)-\frac{d}{dx}[\lambda(x)g'(x)]+\frac{g(a)}{a-x}+\frac{g(-a)}{a+x}. \tag{440}$$

Substituting (440) into (438), we will find

$$\pi^2 \frac{b_0+b_1x+\ldots+b_nx^n}{a_0+a_1x+\ldots+a_nx^n}\frac{g(x)-F(x)}{\sqrt{a^2-x^2}}=$$
$$=f'(x)-\frac{d}{dx}[\lambda(x)g'(x)]+\frac{g(a)}{a-x}+\frac{g(-a)}{a+x}+$$
$$+\frac{\beta_0+\beta_1x+\ldots+\beta_{n-1}x^{n-1}}{a_0+a_1x+\ldots+a_nx^n},$$

or, according to (394),

$$\pi^2\frac{g(x)-F(x)}{\lambda(x)}=$$
$$=f'(x)-\frac{d}{dx}[\lambda(x)g'(x)]+\frac{g(a)}{a-x}+\frac{g(-a)}{a+x}+S(x), \qquad (441)$$

where

$$S(x)=\frac{\beta_0+\beta_1x+\ldots+\beta_{n-1}x^{n-1}}{a_0+a_1x+\ldots+a_nx^n}. \qquad (442)$$

Thus, the solution of equation (432) is reduced by us to the integration of the differential equation (441). Assuming

$$v(x)=\pi\int_0^x\frac{dt}{\lambda(t)}, \qquad (443)$$

we will have

$$\lambda(x)=\pi\frac{dx}{dv}, \quad \frac{1}{\pi}\lambda(x)g'(x)=\frac{dg}{dx}\frac{dx}{dv}=\frac{dg}{dv},$$
$$\frac{1}{\pi^2}\lambda(x)\frac{d}{dx}[\lambda(x)g'(x)]=\frac{d}{dx}\left(\frac{dg}{dv}\right)\frac{dx}{dv}=\frac{d^2g}{dv^2}.$$

Thus, multiplying both sides of equation (441) by $\frac{1}{\pi^2}\lambda(x)$, we will obtain equation

$$\frac{d^2g}{dv^2}+g=\frac{1}{\pi^2}\lambda(x)\left[f'(x)+S(x)+\frac{g(a)}{a-x}+\frac{g(-a)}{a+x}\right]+F(x),$$

or

$$\frac{d^2g}{dv^2}+g=\Psi(x), \qquad (444)$$

where

$$\Psi(x)=\frac{1}{\pi^2}\lambda(x)\left[f'(x)+S(x)+\frac{g(a)}{a-x}+\frac{g(-a)}{a+x}\right]+F(x). \qquad (445)$$

As we already showed above, the differential equation (417) has the solution (427). Consequently, the solution of the differential equation (444) will have the form

$$g=\int_0^v\bar{\Psi}(t)\sin(v-v^*)\,dv^*+c_1\cos v+c_2\sin v, \qquad (446)$$

where variables t and ν^* according to (443) are connected by the relation

$$\nu^*(t) = \pi \int_0^t \frac{dt}{\lambda(t)}. \tag{447}$$

Passing in (446) from variable ν^* to variable t, we will find in accordance with relations (443) and (447)

$$g(x) = \pi \int_0^x \Psi(t) \sin[\nu(x) - \nu(t)] \frac{dt}{\lambda(t)} + c_1 \cos \nu(x) + c_2 \sin \nu(t). \tag{448}$$

Substituting (463) into (466), we will obtain finally

$$\begin{aligned} g(x) = \frac{1}{\pi} \int_0^x \left[\pi^2 \frac{F(t)}{\lambda(t)} + f'(t) + S(t) + \frac{g(a)}{a-t} + \frac{g(-a)}{a+t}\right] \times \\ \times \sin[\nu(x) - \nu(t)]\, dt + c_1 \cos \nu(x) + c_2 \sin \nu(t), \end{aligned} \tag{449}$$

where functions $F(x)$, $S(x)$ and $\nu(x)$ are determined by formulas (446), (442) and (443).

Assuming in (449) $x = a$ and $x = -a$ we will obtain two equations for the determination of constants c_1 and c_2:

$$\left.\begin{aligned} & c_1 \cos \nu(a) + c_2 \sin \nu(a) = \\ & = g(a) - \frac{1}{\pi} \int_0^a \left[\pi^2 \frac{F(t)}{\lambda(t)} + f'(t) + S(t) + \frac{g(a)}{a-t} + \frac{g(-a)}{a+t}\right] \times \\ & \qquad \times \sin[\nu(a) - \nu(t)]\, dt, \\ & c_1 \cos \nu(-a) + c_2 \sin \nu(-a) = \\ & = g(-a) - \frac{1}{\pi} \int_0^{-a} \left[\pi^2 \frac{F(t)}{\lambda(t)} + f'(t) + S(t) + \frac{g(a)}{a-t} + \frac{g(-a)}{a+t}\right] \times \\ & \qquad \times \sin[\nu(-a) - \nu(t)]\, dt. \end{aligned}\right\} \tag{450}$$

Equations (439) by means of partial integration can be given the form

$$\begin{aligned} \int_{-a}^{a} g(t) \frac{d}{dt}\left[\frac{P_k(t)}{b_0 + b_1 t + \ldots + b_n t^n}\right] dt = g(a) \frac{P_k(a)}{b_0 + b_1 a + \ldots + b_n a^n} - \\ - g(-a) \frac{P_k(-a)}{b_0 - b_1 a + \ldots + b_n (-a)^n} - \beta_k, \quad k = 0, 1, \ldots, n-1. \end{aligned}$$

Substituting (449) into (450), we will obtain n equations for the determination of constants $\beta_0, \beta_1, \ldots, \beta_{n-1}$, which appear in the expression for the rational function $S(x)$.

Boundary values of the unknown function $g(a)$ and $g(-a)$, which appear in formula (449), are determined by supplemenatry conditions resulting from the statement of a certain problem, which leads to equation (432).

CHAPTER II

TWO-DIMENSIONAL CONTACT PROBLEM

§ 1. Derivation of Fundamental Equation of the Two-Dimensional Contact Problem

Let us assume that two touching elastic bodies (I and II on Fig. 5) before compression are bounded in the section on plane xOy by curves

$$y = f_1(x) \text{ and } y = -f_2(x). \tag{1}$$

Prior to the compression between the elastic bodies there will be the clearance $f_1(x)+f_2(x)$. Contact of the bodies will take place for those points of the axis Ox, where

$$f_1(x)+f_2(x)=0.$$

The set of points of the axis Ox for which contact of the bodies takes place before compression will be designated by S_0. With compression by forces parallel to the axis Oy, between these elastic bodies generally contact even along certain additional sections of the axis Ox will appear. The set of points of axis Ox for which there is contact between the compressed bodies will be designated by S. In the process of compression the elastic bodies will obtain forward displacement in the direction of axis Oy, which will be designated by $-\alpha_1$ and α_2. Thus, between the compressible bodies an approach α equal to $\alpha_1 + \alpha_2$ will occur. Let us assume that two

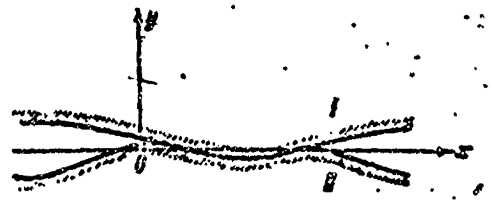

Fig. 5.

points of elastic bodies, which occupied before compression position A_1 and A_2, touched as a result of compression at point A (see Fig. 6 where the dashed line shows the outlines of compressible bodies prior to compression and solid line, after compression). Displacements of these points A_1A and A_2A will consist of forward displacements $A_1'A$ and $A_2'A$ equal respectively to $-\alpha_1$ and α_2 and elastic displacements A_1A_1' and A_2A_2'. Let us designate by u_1, v_1 and $-u_2$, $-v_2$ elastic displacements of points A_1 and A_2 in the direction of axes Ox and Oy. If point A has the abscissa x, then abscissas of points A_1 and A_2 will be respectively equal to $x - u_1$ and $x + u_2$, and ordinates equal to $f_1(x - u_1)$ and $-f_2(x + u_2)$ according to (1). By examining the displacement A_1A of point A_1, we will find for the ordinate of point A the expression

$$f_1(x-u_1)+v_1-\alpha_1;$$

by examining the displacement A_2A of point A_2, we will obtain for the ordinate of point A the value

$$-f_2(x+u_2)-v_2+\alpha_2.$$

Thus, the equality

$$f_1(x-u_1)+v_1-\alpha_1=-f_2(x+u_2)-v_2+\alpha_2,$$

or

$$v_1+v_2=\alpha-f_1(x-u_1)-f_2(x+u_2), \qquad (2)$$

should take place where $\alpha = \alpha_1 + \alpha_2$ is the approach of elastic bodies with compression. By examining only small elastic displacements, we

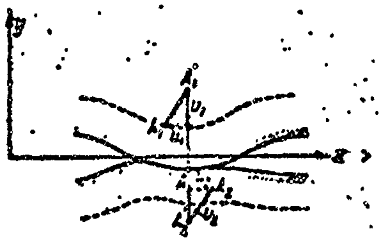

Fig. 6.

can replace in (2) $f_1(x - u_1)$ and $f_2(x + u_2)$ by $f_1(x)$ and $f_2(x)$. Let us obtain then for points of contact the condition

$$v_1 + v_2 = \alpha - f_1(x) - f_2(x) \text{ on } S. \qquad (3)$$

We will further assume that the friction between the compressible bodies is absent. Then at the points of contact each of the compressed bodies will undergo on the side of the other body only normal pressure, which we will designate by $p(x)$. Assuming that the whole region of contact is small in comparison with dimensions of the compressible bodies, we will consider that elastic displacements v_1 and v_2 at the point with the abscissa x will be the same as those at boundary points of two elastic half-planes (upper and lower), which are under the impact of the same normal pressure $p(x)$ as that of the examined compressible bodies.

Let us examine the lower elastic half-plane to boundary of which is applied normal pressure $p(x)$ on sections of the axis Ox, which correspond to sections of contact of the compressible bodies (Fig. 7a). Let us separate on any of these sections the segment of the axis Ox from point $x = t$ (Fig. 7b) up to an infinitely close point $x = t + dt$. On this section force $p(t)dt$ will act. Since section dt, on which this force acts, is infinitesimal, the action of this force on the elastic half-plane will be the same as if to the elastic half-plane an infinitesimal concentrated force $p(t)dt$ were applied at point $x = t$. The problem about the action of a normal concentrated force on the boundary of the elastic half-plane is well-known in the theory of elasticity[1]. If at point $x = t$ to

[1]See Timoshenko, S. P. Theory of elasticity, ONTI 1937, p. 101.

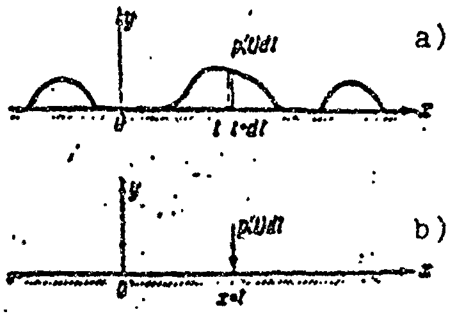

Fig. 7.

the boundary of the elastic half-plane a normal concentrated force P is applied, then the end point of the elastic medium with abscissa x obtains displacement v in the direction of axis Oy equal to

$$v = -\vartheta P \ln \frac{1}{r} + \text{const.}, \tag{4}$$

where

$$r = |t - x| \tag{5}$$

is the distance between points of the axis Ox with abscissas t and x;

$$\vartheta = \frac{2}{\pi E}(1 - \mu^2), \tag{6}$$

where E is the elastic modulus, μ – Poisson's ratio.

Thus, force $p(t)dt$, applied to the boundary of the elastic half-plane at point $t = x$, will cause at the point of the boundary with abscissa x displacement in the direction of the Oy axis:

$$dv = -\vartheta p(t) \ln \frac{1}{|t - x|} dt + \text{const.},$$

and the action of the whole load p, applied (Fig. 7a) to the boundary of the elastic half-plane, will create at the point with the abscissa x the displacement

$$v = -\vartheta \int_S p(t) \ln \frac{1}{|t-x|}\, dt + \text{const.}^{1} \qquad (7)$$

If the same normal pressure p will act on the boundary of the upper elastic half-plane, then the end point with the abscissa x will obtain displacement v in the direction of the Ox axis equal to

$$v = \vartheta \int_S p(t) \ln \frac{1}{|t-x|}\, dt + \text{const.} \qquad (8)$$

Thus under the made assumptions displacements v_1 and v_2 in Fig. 6 will respectively equal

$$v_1 = \vartheta_1 \int_S p(t) \ln \frac{1}{|t-x|}\, dt + \text{const.} \qquad (9)$$

according to the formula (8) for the upper half-plane and

$$-v_2 = -\vartheta_2 \int_S p(t) \ln \frac{1}{|t-x|}\, dt + \text{const.} \qquad (10)$$

according to the formula (7) for the lower half-plane. In formulas (9) and (10)

$$\vartheta_1 = \frac{2}{\pi E_1}(1-\mu_1^2), \quad \vartheta_2 = \frac{2}{\pi E_2}(1-\mu_2^2), \qquad (11)$$

where E_1 and μ_1 – elastic constants of the first body, and E_2 and μ_2 – elastic constants of the second body.

Substituting v_1 and v_2 from (9) and (10) into (3), we will obtain for pressure $p(x)$ the integral equation

$$(\vartheta_1 + \vartheta_2) \int_S p(t) \ln \frac{1}{|t-x|}\, dt = c - f_1(x) - f_2(x) \text{ on } S, \qquad (12)$$

---

[1]The problem of the action of the concentrated force on the elastic medium should be examined as an abstraction not reflecting practically possible conditions of the problem of the theory of of elasticity. However, in using this formal solution of equations of the theory of elasticity, it is easy to turn to the solution of the real problem about the action of the continuously distributed load on the elastic medium.

or

$$\int_S \mu(t)\ln\frac{1}{|t-x|}\,dt = f(x) \text{ on } S, \tag{13}$$

where

$$f(x) = \frac{c - f_1(x) - f_2(x)}{\theta_1+\theta_2}, \tag{14}$$

c is a certain constant. Equation (13) is the basic integral equation of a two-dimensional contact problem of the theory of elasticity and it is examined in detail in Chapter I.

§ 2. The Case of One Section of Compression of Elastic Bodies

Let us examine first the case when the initial contact of compressible bodies in plane xOy occurs at one point. Let us take this point as the origin of the coordinates (Fig. 8). We will first assume that functions $f_1(x)$ and $f_2(x)$, which determine the configuration of the compressible bodies, have continuous first and second derivatives in the neighborhood of point $x = 0$. Directing axis Ox along a common tangent to the curves limiting elastic bodies in plane xOy, we will have

$$f_1'(0) = f_2'(0) = 0. \tag{15}$$

The sum of the second derivatives

$$f_1''(0) + f_2''(0)$$

will at first be assumed to be different from zero. In view of smallness of the elastic displacements, the region of contact S after compression of the elastic bodies will be small, and in this region the sum of functions $f_1(x)+f_2(x)$ will be approximately possible to represent in the form

$$f_1(x)+f_2(x) = \frac{1}{2}[f_1''(0)+f_2''(0)]x^2. \tag{16}$$

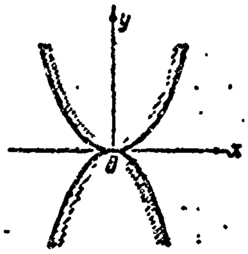

Fig. 8.

With respect to forces compressing body, we will consider that their resultants, perpendicular Ox axis, are directed to the point of initial contact of compressible bodies, i.e., to the origin of the coordinates. Since we assume the initial opening between compressible bodies $f_1(x) + f_2(x)$ according to (16) to be symmetric with respect to the Oy axis, pressure p on surfaces of compressed bodies will also be symmetric with respect to the Oy axis. The region of contact between compressed bodies S will constitute a certain segment of the Ox $-a<x<a$ axis. The integral equation (13) will have the form

$$\int_{-a}^{a} p(t)\ln\frac{1}{|t-x|}\,dt = f(x) \text{ when } -a<x<a, \tag{17}$$

where according to (14) and (16)

$$f(x) = \frac{c-\frac{1}{2}[f_1''(0)+f_2''(0)]x^2}{\vartheta_1+\vartheta_2};$$

or

$$f(x) = \alpha - Ax^2, \tag{18}$$

where

$$A = \frac{f_1''(0)+f_2''(0)}{2(\vartheta_1+\vartheta_2)}, \tag{19}$$

and α — certain constant.

The integral equation (17) coincides with equation (1), examined by us in detail in §§ 1, 2 and 3 of Chapter I. For the case when the right side of this equation $f(x)$ has the form (18), the solution

$$p(x)=\frac{P+Aa^2-2Ax^2}{\pi\sqrt{a^2-x^2}}\tag{20}$$

was found by us (formula (77) of Chapter I), where

$$P=\int_{-a}^{a}p(x)\,dx.\tag{21}$$

Relation (21) shows that constant P, which enters into formula (20), determines the resultant of the compressing forces applied to each of the compressed bodies and balanced by the pressure acting on surface of pressure. We will consider force P assigned. It remains to determine half-width of the section of contact a, which enters into formula (20). It is determined by the condition that pressure $p(x)$ should be limited everywhere, including the edge of the section of contact. This is possible only when

$$P=Aa^2,\tag{22}$$

and formula (20) takes the form

$$p(x)=\frac{2P}{\pi a^2}\sqrt{a^2-x^2}.\tag{23}$$

Substituting (19) into (22), we will find

$$a=\sqrt{\frac{2P(\vartheta_1+\vartheta_2)}{f_1''(0)+f_2''(0)}}.\tag{24}$$

Formulas (23) and (24) completely solve the problem by determining according to the compressing force P the half-width of the section of contact a and pressure in the region of contact $p(x)$.

Let us examine now that special case when the sum of the second derivatives is determined by the relation

$$f_1''(0)+f_2''(0)=0.\tag{25}$$

For generality we will assume that not only the second derivative of the sum $f_1(x)+f_2(x)$ but also all subsequent derivatives up to

the $(2n - 1)$-th, inclusively, turn into zero when $x = 0$, and derivative

$$f_1^{(2n)}(x)+f_2^{(2n)}(x)$$

is different from zero when $x = 0$, being continuous in this point. In this case, considering the smallness of the section of contact, when $-a<x<a$ we can approximately substitute

$$f_1(x)+f_2(x)=\frac{1}{(2n)!}[f_1^{(2n)}(0)+f_2^{(2n)}(0)]\,x^{2n}. \qquad (26)$$

Substituting (26) into (14), we will find that in this case

$$f(x)=\alpha-A_n x^{2n}, \qquad (27)$$

where

$$A_n=\frac{f_1^{(2n)}(0)+f_2^{(2n)}(0)}{(2n)!\,(\vartheta_1+\vartheta_2)}, \qquad (28)$$

and α – certain constant.

In order to solve the integral equation (17) for the case when the right side of this equation has the form of (27), let us use the general formula for the solution of equation (17):

$$p(x)=\frac{1}{\pi\sqrt{a^2-x^2}}\left[P-\frac{1}{\pi}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{t-x}\,dt\right] \qquad (29)$$

(see formula (115) of Chapter I).

In order that function $p(x)$, determined by formula (29), remains limited when $x = a$, condition

$$P=-\frac{1}{\pi}\int_{-a}^{a}\frac{f'(t)\sqrt{a^2-t^2}}{a-t}\,dt, \qquad (30)$$

should be fulfilled. Substituting (30) into (29) and using identity

$$\frac{1}{a-t}+\frac{1}{t-x}=\frac{a^2-x^2}{(a^2-t^2)(t-x)}+\frac{a-x}{a^2-t^2},$$

we will obtain

$$p(x) = -\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}(t-x)} - \frac{1}{\pi^2}\sqrt{\frac{a-x}{a+x}}\int_{-a}^{a}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}}. \tag{31}$$

So that function $p(x)$, determined by formula (31), remains limited when $x = -a$, condition

$$\int_{-a}^{a}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}} = 0. \tag{32}$$

should be fulfilled. If condition (32) is fulfilled, then, using identity

$$\frac{\sqrt{a^2-t^2}}{a-t} = \frac{a+t}{\sqrt{a^2-t^2}},$$

formula (30) can be given the form

$$\int_{-a}^{a}\frac{f'(t)\,t\,dt}{\sqrt{a^2-t^2}} = -\pi P, \tag{33}$$

and formula (31) will in this case have the form

$$p(x) = -\frac{1}{\pi^2}\sqrt{a^2-x^2}\int_{-a}^{a}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}(t-x)}. \tag{34}$$

For the case when function $f(x)$ has the form (27), condition (32) is fulfilled, since in this case in the definite integral (32) the integrand will be odd. Substituting (27) into (34), we will find

$$p(x) = \frac{2n}{\pi^2}A_n\sqrt{a^2-x^2}\int_{-a}^{a}\frac{t^{2n-1}\,dt}{\sqrt{a^2-t^2}(t-x)}. \tag{35}$$

Using identity

$$\frac{t^{2n-1}}{t-x} = \frac{x^{2n-1}}{t-x} + t^{2n-2} + t^{2n-3}x + \ldots + tx^{2n-3} + x^{2n-2},$$

formula (35) can be given the form

$$p(x)=\frac{2n}{a^2}A_n\sqrt{a^2-x^2}\times$$
$$\times\left[\sum_{m=0}^{2n-2}J_m a^m x^{2n-2-m}+x^{2n-1}\int_{-a}^{a}\frac{dt}{\sqrt{a^2-t^2}\,(t-x)}\right], \qquad (36)$$

where

$$J_m=\frac{1}{a^m}\int_{-a}^{a}\frac{t^m\,dt}{\sqrt{a^2-t^2}}. \qquad (37)$$

Integrating by parts, we will find

$$\frac{1}{a^m}\int_{-a}^{a}\frac{t^m\,dt}{\sqrt{a^2-t^2}}=-\frac{t^{m-1}}{a^m}\sqrt{a^2-t^2}\Big|_{t=-a}^{t=+a}+\frac{m-1}{a^m}\int_{-a}^{a}t^{m-2}\sqrt{a^2-t^2}\,dt=$$
$$=\frac{m-1}{a^{m-2}}\int_{-a}^{a}\frac{t^{m-2}\,dt}{\sqrt{a^2-t^2}}-\frac{m-1}{a^m}\int_{-a}^{a}\frac{t^m\,dt}{\sqrt{a^2-t^2}},$$

whence, using designation (37), we will find

$$J_m=\frac{m-1}{m}J_{m-2}; \qquad (38)$$

When m is even, from formula (38) it follows that

$$J_m=\frac{m-1}{m}J_{m-2}=\frac{(m-1)(m-3)}{m(m-2)}J_{m-4}=\ldots=\frac{(m-1)(m-3)\ldots3\cdot1}{m(m-2)\ldots4\cdot2}J_0=$$
$$=\frac{1\cdot3\ldots(m-3)(m-1)}{2\cdot4\ldots(m-2)m}\int_{-a}^{a}\frac{dt}{\sqrt{a^2-t^2}}=\frac{1\cdot3\ldots(m-3)(m-1)}{2\cdot4\ldots(m-2)m}\pi, \qquad (39)$$

When m is odd, $J_m = 0$, since in this case the integrand in (37) is odd.

In Chapter I we showed that equation

$$\int_{-a}^{a}p(t)\ln\frac{1}{|t-x|}\,dt=\alpha,\quad -a<x<a, \qquad (40)$$

where α – constant, has the solution

$$p(x)=\frac{P}{\pi\sqrt{a^2-x^2}} \qquad (41)$$

(see formula (74) of Chapter I). Substituting (41) into (40), we will find

$$\int_{-a}^{a}\frac{1}{\sqrt{a^2-t^2}}\ln\frac{1}{|t-x|}\,dt=\text{const. when } -a<x<a. \qquad (42)$$

Hence, differentiating with respect to x, we will find

$$\int_{-a}^{a}\frac{dt}{\sqrt{a^2-t^2}\,(t-x)}=0 \text{ when } -a<x<a. \qquad (43)$$

Substituting (39) and (43) into (36), we will obtain

$$p(x)=\frac{2n}{\pi}A_n\sqrt{a^2-x^2}\left[\frac{1\cdot 3\ldots(2n-3)}{2\cdot 4\ldots(2n-2)}a^{2n-2}+\right.$$
$$\left.+\frac{1\cdot 3\ldots(2n-5)}{2\cdot 4\ldots(2n-4)}a^{2n-4}x^2+\ldots+\frac{1}{2}a^2x^{2n-4}+x^{2n-2}\right]. \qquad (44)$$

Substituting (27) into (33), we will find

$$2nA_n\int_{-a}^{a}\frac{t^{2n}\,dt}{\sqrt{a^2-t^2}}=\pi P,$$

or, using designation (37),

$$2nA_na^{2n}J_{2n}=\pi P.$$

Hence, according to (39)

$$\frac{1\cdot 3\ldots(2n-3)(2n-1)}{2\cdot 4\ldots(2n-4)(2n-2)}A_na^{2n}=P. \qquad (45)$$

Substituting A_n from (45) in (44), we will find

$$p(x)=\frac{P}{\pi a^2}\sqrt{a^2-x^2}\left[\frac{2n}{2n-1}+\frac{2n(2n-2)}{(2n-1)(2n-3)}\frac{x^2}{a^2}+\right.$$
$$\left.+\ldots\frac{2n(2n-2)\ldots 2}{(2n-1)(2n-3)\ldots 3\cdot 1}\frac{x^{2n-2}}{a^{2n-2}}\right]. \qquad (46)$$

Substituting (28) into (45), we will find

$$a=\sqrt[2n]{\frac{2\cdot 4\ldots(2n-4)(2n-2)(2n)(\vartheta_1+\vartheta_2)P}{1\cdot 3\ldots(2n-3)(2n-1)\left[f_1^{(2n)}(0)+f_2^{(2n)}(0)\right]}},$$

or

$$a = 2\cdot 4\ldots(2n-2)2n \sqrt[2n]{\frac{(\vartheta_1+\vartheta_2)P}{2n[f_1^{(2n)}(0)+f_2^{(2n)}(0)]}}. \qquad (47)$$

Formulas (46) and (47) determine the half-width of the section of contact a and pressure in the region of contact $p(x)$. When $n = 1$ formulas (46) and (47) turn into formulas (23) and (24). Figure 9 shows[1] graphs of function $p(x)$ for different n, which correspond to the identical half-width of the section of contact a and identical compressing force P[2].

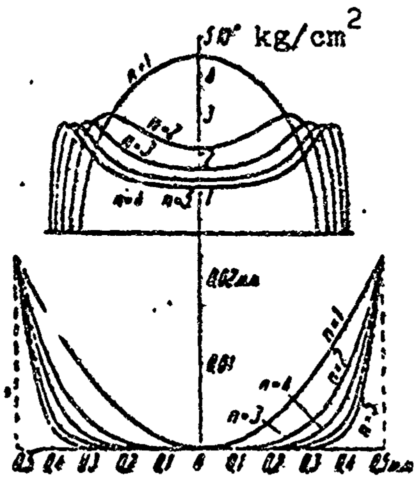

Fig. 9.

Thus far we assumed that the second derivative of the sum of functions $f_1(x)+f_2(x)$ is continuous in the neighborhood of point $x = 0$. Let us consider now the special case when point $x = 0$ is the point of discontinuity for the second derivative of the sum of functions $f_1(x)+f_2(x)$. In this case the indicated second derivative can, either by remaining limited, have a jump at point $x = 0$, or turn into infinity at this point.

[1]See also my article in Reports of the Academy of Sciences of the USSR, Vol. 25, No. 5, 1936.

[2]In calculations $E = 2\cdot 10^6$ kg/cm^2 and $\sigma = 0.3$ are accepted.

Let us start from the first of these two cases. Thus, let us assume that

$$\left.\begin{array}{l} f_1'(0)+f_2'(0)=2A_+ \text{ when } x=+0, \\ f_1'(0)+f_2'(0)=2A_- \text{ when } x=-0. \end{array}\right\} \tag{48}$$

Then, considering the smallness of the section of contact, it is possible approximately to substitute

$$\left.\begin{array}{l} f_1(x)+f_2(x)=A_+x^2 \text{ when } x>0, \\ f_1(x)+f_2(x)=A_-x^2 \text{ when } x<0. \end{array}\right\} \tag{49}$$

We will first consider, in accordance with initial assumptions of § 1 of this chapter, that compressible bodies can have only forward displacements parallel to the Oy axis. In this case we can use for the determination of pressure $p(x)$ equation (13), the right side of which $f(x)$ is determined by relation (14). The region of contact of compressed bodies S will not now be symmetric relative to the origin of the coordinates. Let us designate the abscissas of the beginning and end of the section of contact by $-a + \delta$ and $a + \delta$. Then equation (13) will have the form

$$\int_{-a+\delta}^{a+\delta} p(t)\ln\frac{1}{|t-x|}\,dt=f(x), \quad -a+\delta\leqslant x\leqslant a+\delta. \tag{50}$$

Assuming in (50)

$$t=\tau+\delta, \quad x=\xi+\delta, \tag{51}$$

we will obtain equation

$$\int_{-a}^{a} p(\tau+\delta)\ln\frac{1}{|\tau-\xi|}\,d\tau=f(\xi+\delta), \quad -a<\xi<a, \tag{52}$$

which according to the formula (34) will have the solution

$$p(\xi+\delta)=-\frac{1}{\pi^2}\sqrt{a^2-\xi^2}\int_{-a}^{a}\frac{f'(\tau+\delta)\,d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)}, \tag{53}$$

where conditions (32) and (33) should be fulfilled, which in this case will have the form

$$\int_{-a}^{a} \frac{f'(\tau+\delta)\,d\tau}{\sqrt{a^2-\tau^2}} = 0, \quad \int_{-a}^{a} \frac{f'(\tau+\delta)\,\tau\,d\tau}{\sqrt{a^2-\tau^2}} = -\pi P. \tag{54}$$

Substituting (49) into (14), we will find

$$\left.\begin{aligned} f'(x) &= -\frac{2A_+}{\vartheta_1+\vartheta_2}\,x \quad \text{when } x>0, \\ f'(x) &= -\frac{2A_-}{\vartheta_1+\vartheta_2}\,x \quad \text{when } x<0. \end{aligned}\right\} \tag{55}$$

Substituting (55) into (53), we will obtain

$$\begin{aligned} p(\xi+\delta) &= \frac{2\sqrt{a^2-\xi^2}}{\pi^2(\vartheta_1+\vartheta_2)}\left[A_-\int_{-a}^{-\delta}\frac{(\tau+\delta)\,d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} + A_+\int_{-\delta}^{a}\frac{(\tau+\delta)\,d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)}\right] = \\ &= \frac{2\sqrt{a^2-\xi^2}}{\pi^2(\vartheta_1+\vartheta_2)}\left\{A_-\int_{-a}^{-\delta}\frac{d\tau}{\sqrt{a^2-\tau^2}} + A_+\int_{-\delta}^{a}\frac{d\tau}{\sqrt{a^2-\tau^2}} + \right. \\ &\left. + (\xi+\delta)\left[A_-\int_{-a}^{-\delta}\frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} + A_+\int_{-\delta}^{a}\frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)}\right]\right\}. \end{aligned} \tag{56}$$

Assuming

$$\tau = \frac{2au}{1+u^2}, \quad \xi = \frac{2av}{1+v^2}, \quad \delta = \frac{2au_0}{1+u_0^2}, \tag{57}$$
$$(|u|<1 \text{ and } |v|<1 \text{ when } |\tau|<a \text{ and } |\xi|<a),$$

we will find

$$\begin{aligned} \int_{-a}^{-\delta}\frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} &= \frac{1+v^2}{a}\int_{-1}^{-u_0}\frac{du}{(u-v)(1-uv)} = \\ &= \frac{1+v^2}{a(1-v^2)}\left(\int_{-1}^{-u_0}\frac{du}{u-v} + \int_{-1}^{-u_0}\frac{du}{\frac{1}{v}-u}\right). \end{aligned} \tag{58}$$

When $\xi < -\delta$, $v < -u_0$, by calculating the principal value of the definite integral, we will find

$$\begin{aligned} \int_{-1}^{-u_0}\frac{du}{u-v} &= \lim_{\varepsilon\to 0}\left(\int_{-1}^{v-\varepsilon}\frac{du}{u-v} + \int_{v+\varepsilon}^{-u_0}\frac{du}{u-v}\right) = \lim_{\varepsilon\to 0}\left[\ln(v-u)\Big|_{u=-1}^{u=v-\varepsilon} + \right. \\ &\left. + \ln(u-v)\Big|_{u=v+\varepsilon}^{u=-u_0}\right] = \lim_{\varepsilon\to 0}\left[\ln\frac{\varepsilon(-u_0-v)}{(1+v)\,\varepsilon}\right] = \ln\left(-\frac{u_0+v}{1+v}\right). \end{aligned} \tag{59}$$

When $\xi > -\delta$, $v > -u_0$, we find directly

$$\int_{-1}^{-u_0} \frac{du}{u-v} = \ln(v-u)\Big|_{u=-1}^{u=-u_0} = \ln\frac{u_0+v}{1+v}. \tag{60}$$

Formulas (59) and (60) can be united into one:

$$\int_{-1}^{-u_0} \frac{du}{u-v} = \ln\frac{|u_0+v|}{1+v}. \tag{61}$$

We find further

$$\int_{-1}^{-u_0} \frac{du}{\frac{1}{v}-u} = -\ln\left(\frac{1}{v}-u\right)\Big|_{u=-1}^{u=-u_0} = \ln\frac{1+v}{1+u_0 v}. \tag{62}$$

Substituting (61) and (62) into (58), we will find

$$\int_{-a}^{-\delta} \frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} = \frac{1+v^2}{a(1-v^2)}\ln\frac{|u_0+v|}{1+u_0 v} \qquad (|v|<1). \tag{63}$$

Considering further

$$v=\mathrm{tg}\left(\frac{\pi}{4}-\frac{\varphi}{2}\right),\ u_0=\mathrm{tg}\left(\frac{\varphi_0}{2}-\frac{\pi}{4}\right) \quad (0<\varphi<\pi,\ 0<\varphi_0<\pi), \tag{64}$$

we will find according to (57)

$$\xi = a\cos\varphi, \quad \delta = -a\cos\varphi_0; \tag{65}$$

and formula (63) will take in this case the form

$$\int_{-a}^{-\delta} \frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} = \frac{1}{a\sin\varphi}\ln\left|\frac{\sin\frac{\varphi-\varphi_0}{2}}{\sin\frac{\varphi+\varphi_0}{2}}\right|. \tag{66}$$

According to (43)

$$\int_{-\delta}^{a} \frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} = -\int_{-a}^{-\delta} \frac{d\tau}{\sqrt{a^2-\tau^2}\,(\tau-\xi)} = \frac{1}{a\sin\varphi}\ln\left|\frac{\sin\frac{\varphi+\varphi_0}{2}}{\sin\frac{\varphi-\varphi_0}{2}}\right|. \tag{67}$$

Assuming $\tau = a\cos\varphi$ and taking into account (65), we will find

$$\left.\begin{aligned}\int_{-a}^{-\delta}\frac{d\tau}{\sqrt{a^2-\tau^2}}&=\int_{\varphi_0}^{\pi}d\varphi=\pi-\varphi_0,\\ \int_{-\delta}^{a}\frac{d\tau}{\sqrt{a^2-\tau^2}}&=\int_{0}^{\varphi_0}d\varphi=\varphi_0.\end{aligned}\right\} \tag{68}$$

Substituting (65), (66), (67) and (68) into (56), we will find

$$p=\frac{2a}{\pi^2(\vartheta_1+\vartheta_2)}\left\{\sin\varphi\,[A_+\varphi_0+A_-(\pi-\varphi_0)]+\right.$$
$$\left.+(\cos\varphi-\cos\varphi_0)(A_+-A_-)\ln\left|\frac{\sin\frac{\varphi+\varphi_0}{2}}{\sin\frac{\varphi-\varphi_0}{2}}\right|\right\}, \tag{69}$$
$$0<\varphi<\pi.$$

According to (51) and (65)

$$x=a(\cos\varphi-\cos\varphi_0). \tag{70}$$

Formulas (69) and (70) determine function $p(x)$ in the interval $-a+\delta<x<a+\delta$.

Substituting (55) into (54), we will obtain the equation

$$\left.\begin{aligned}&A_-\int_{-a}^{-\delta}\frac{(\tau+\delta)\,d\tau}{\sqrt{a^2-\tau^2}}+A_+\int_{-\delta}^{a}\frac{(\tau+\delta)\,d\tau}{\sqrt{a^2-\tau^2}}=0,\\ &A_-\int_{-a}^{-\delta}\frac{(\tau+\delta)\,\tau\,d\tau}{\sqrt{a^2-\tau^2}}+A_+\int_{-\delta}^{a}\frac{(\tau+\delta)\,\tau\,d\tau}{\sqrt{a^2-\tau^2}}=\frac{\pi}{2}P(\vartheta_1+\vartheta_2).\end{aligned}\right\} \tag{71}$$

Assuming in (71) $\tau=a\cos\varphi$, $\delta=-a\cos\varphi_0$, we will find

$$A_-\int_{\varphi_0}^{\pi}(\cos\varphi-\cos\varphi_0)\,d\varphi+A_+\int_{0}^{\varphi_0}(\cos\varphi-\cos\varphi_0)\,d\varphi=0,$$
$$a^2\left[A_-\int_{\varphi_0}^{\pi}(\cos\varphi-\cos\varphi_0)\cos\varphi\,d\varphi+A_+\int_{0}^{\varphi_0}(\cos\varphi-\cos\varphi_0)\cos\varphi\,d\varphi\right]=$$
$$=\frac{\pi}{2}P(\vartheta_1+\vartheta_2)$$

or, fulfilling integration

$$\left.\begin{aligned} A_0(\sin\varphi_0-\varphi_0\cos\varphi_0)-A_-(\sin\varphi_0-\varphi_0\cos\varphi_0+\pi\cos\varphi_0)=0,\\ a^2\{A_0(\varphi_0-\sin\varphi_0\cos\varphi_0)-A_-(\varphi_0-\sin\varphi_0\cos\varphi_0-\pi)\}=\\ =\pi P(\vartheta_1+\vartheta_2). \end{aligned}\right\} \qquad (72)$$

The first of equations (72) gives

$$\operatorname{tg}\varphi_0-\varphi_0=\frac{\pi}{\frac{A_0}{A_-}-1}. \qquad (73)$$

Multiplying the first of equations (72) by $\frac{a^2}{\cos\varphi_0}$ and adding it to the second, we will find

$$a^2\sin^2\varphi_0\operatorname{tg}\varphi_0(A_0-A_-)=\pi P(\vartheta_1+\vartheta_2),$$

whence

$$a=\frac{1}{\sin\varphi_0}\sqrt{\frac{\pi P(\vartheta_1+\vartheta_2)}{(A_0-A_-)\operatorname{tg}\varphi_0}}. \qquad (74)$$

Having determined ϕ_0 from equation (73), from formula (74) we will find the half-width of the section of contact a, by the second of formulas (65) – the displacement of the section of contact with respect to the origin of coordinates δ, and further, by formulas (69) and (70), we will be able to determine pressure $p(x)$ in the region of contact $-a + \delta < x < a + \delta$.

Figure 10 shows the graph of pressure $p(x)$ for the case $\varphi_0=120°$. Let us note that pressure $p(x)$ in the examined case is reduced not only to force P, applied at the origin of the coordinates, but also to a certain moment M with respect to the point $x = 0$. Let us calculate this moment M. We find

$$M=\int_{-a+\delta}^{a+\delta}p(t)t\,dt=\int_{-a}^{a}p(\tau+\delta)(\tau+\delta)\,d\tau=$$

$$=P\delta+\int_{-a}^{a}p(\tau+\delta)\tau\,d\tau, \qquad (75)$$

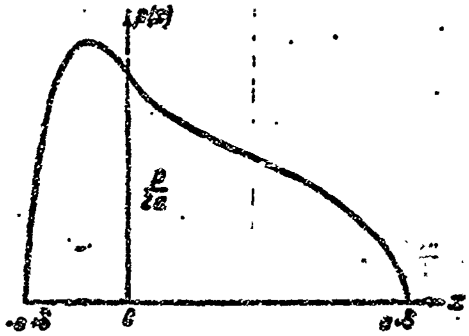

Fig. 10.

since

$$\int_{-a}^{a} p(\tau+\delta)\,d\tau = \int_{-a+\delta}^{a+\delta} p(\xi)\,d\xi = P.$$

Differentiating (52) with respect to ξ, multiplying by $\sqrt{a^2-\xi^2}$ and integrating then with respect to ξ from $-a$ to a, we will find

$$\int_{-a}^{a} p(\tau+\delta)\left[\int_{-a}^{a} \frac{\sqrt{a^2-\xi^2}}{\tau-\xi}\,d\xi\right]d\tau = \int_{-a}^{a} f'(\xi+\delta)\sqrt{a^2-\xi^2}\,d\xi. \qquad (76)$$

Let us find further

$$\int_{-a}^{a} \frac{\sqrt{a^2-\xi^2}}{\tau-\xi}\,d\xi = (a^2-\tau^2)\int_{-a}^{a} \frac{d\xi}{\sqrt{a^2-\xi^2}\,(\tau-\xi)} + \tau\int_{-a}^{a} \frac{d\xi}{\sqrt{a^2-\xi^2}} + \int_{-a}^{a} \frac{\xi\,d\xi}{\sqrt{a^2-\xi^2}} = \tau\int_{-a}^{a} \frac{d\xi}{\sqrt{a^2-\xi^2}} = \pi\tau, \qquad (77)$$

if one were to consider (43). Substituting (77) into (76), we will obtain

$$\int_{-a}^{a} p(\tau+\delta)\,\tau\,d\tau = \frac{1}{\pi}\int_{-a}^{a} f'(\xi+\delta)\sqrt{a^2-\xi^2}\,d\xi. \qquad (78)$$

Substituting (78) into (75) and changing the designation of the argument according to which integration is conducted, we will find

$$M = P\delta + \frac{1}{\pi}\int_{-a}^{a} f'(\tau+\delta)\sqrt{a^2-\tau^2}\,d\tau. \qquad (79)$$

Substituting (55) into (79), we will obtain

$$M=P\delta-\frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(A_-\int_{-a}^{-\delta}(\tau+\delta)\sqrt{a^2-\tau^2}\,d\tau+\right.$$

$$\left.+A_+\int_{-\delta}^{a}(\tau+\delta)\sqrt{a^2-\tau^2}\,d\tau\right),$$

or, if one were to substitute $\tau=a\cos\varphi,\ \delta=-a\cos\varphi_0$,

$$M=P\delta-\frac{2a^3}{\pi(\vartheta_1+\vartheta_2)}\left(A_-\int_{\varphi_0}^{\pi}\sin^2\varphi(\cos\varphi-\cos\varphi_0)\,d\varphi+\right.$$

$$\left.+A_+\int_{0}^{\varphi_0}\sin^2\varphi(\cos\varphi-\cos\varphi_0)\,d\varphi\right)=P\delta-\frac{2a^3}{\pi(\vartheta_1+\vartheta_2)}\left\{\frac{A_+-A_-}{3}\sin^3\varphi_0-\right.$$

$$\left.-\frac{1}{2}\cos\varphi_0\left[(A_+-A_-)(\varphi_0-\sin\varphi_0\cos\varphi_0)+\pi A_-\right]\right\}=P\delta-$$

$$-\frac{2a^3}{\pi(\vartheta_1+\vartheta_2)}\left\{-\frac{A_+-A_-}{6}\sin^3\varphi_0-\frac{1}{2}\cos\varphi_0\left[(A_+-A_-)(\varphi_0-\operatorname{tg}\varphi_0)+\right.\right.$$

$$\left.\left.+\pi A_-\right]\right\}=P\delta+\frac{(A_+-A_-)a^3\sin^3\varphi_0}{3\pi(\vartheta_1+\vartheta_2)}, \qquad (80)$$

on the basis of (73). Substituting into (80) $a^3\sin^3\varphi_0$ from (74), we will find

$$M=P\delta+\frac{a\cos\varphi_0}{3}P=\frac{2}{3}P\delta, \qquad (81)$$

since $a\cos\varphi_0=-\delta$.

Thus, in order that the compressed bodies are in equilibrium, it is necessary that the resultants of compressing forces be displaced with respect to the point of initial contact of the bodies and cross the Ox axis at point $x=\frac{2}{3}\delta$. If according to conditions set by us above, resultants of compressing forces are directed to the origin of the coordinates, but at the same time compressible bodies can accomplish only forward displacements, then the connection preventing the turn of the bodies with compression will take the moment M determined by relation (81).

Let us assume now that resultants of the compressing forces, just as earlier, are directed to the origin of the coordinates, but the connection hindering turns of the compressible bodies is absent. Let us solve the contact problem under these conditions.

The relation (3), which connects elastic displacements of end points of compressible bodies v_1 and v_2, is derived by us in the assumption that with compression the elastic bodies accomplish only forward displacements $-\alpha_1$ and α_2 in the direction of the Oy axis, and between them besides the approach $\alpha = \alpha_1 + \alpha_2$ occurs. Let us assume now that with compression the elastic body, located in the upper half-plane, besides accomplishing forward displacement accomplishes still a turn relative to the origin of the coordinates by angle $-\theta_1$, and the elastic body turns relative to the origin of coordinates at angle θ_2 (we will consider the turns counterclockwise). Then between end points of compressible bodies having the abscissa x an additional approach equal to θx will occur, where $\theta = \theta_1 + \theta_2$. In order to obtain in this case the connection between elastic displaceemtns v_1 and v_2, it is necessary to replace the constant approach α in formula (3) by a variable approach $\alpha + \theta x$. Let us obtain the relation

$$v_1 + v_2 = \alpha + \theta x - f_1(x) - f_2(x) \text{ on } S. \qquad (82)$$

Substituting into (3) v_1 and v_2 from (9) and (10), we will arrive, moreover, at equation (13), but for function $f(x)$ instead of (14) we will obtain expression

$$f(x) = \frac{\alpha + \theta x - f_1(x) - f_2(x)}{\vartheta_1 + \vartheta_2}. \qquad (83)$$

Substituting (83) into (53), we will find

$$p(\xi + \delta) = -\frac{1}{\pi^2} \frac{\sqrt{a^2 - \xi^2}}{\vartheta_1 + \vartheta_2} \left[\theta \int_{-a}^{a} \frac{d\tau}{\sqrt{a^2 - \tau^2}(\tau - \xi)} - \int_{-a}^{a} \frac{f_1'(\tau + \delta) + f_2'(\tau + \delta)}{\sqrt{a^2 - \tau^2}(\tau - \xi)} d\tau \right]. \qquad (84)$$

But the first integral in formula (84) is equal to zero. Thus, the expression for pressure p will remain the same as that in the absence of the relative turn of compressible bodies θ, and as before for pressure p we will have formula (69). Substituting (83) into (54), we will obtain equations

$$\left.\begin{aligned} &\theta\int_{-a}^{a}\frac{d\tau}{\sqrt{a^2-\tau^2}}-\int_{-a}^{a}\frac{f_1'(\tau+\delta)+f_2'(\tau+\delta)}{\sqrt{a^2-\tau^2}}\,d\tau=0,\\ &\theta\int_{-a}^{a}\frac{\tau\,d\tau}{\sqrt{a^2-\tau^2}}-\int_{-a}^{a}\frac{f_1'(\tau+\delta)+f_2'(\tau+\delta)}{\sqrt{a^2-\tau^2}}\,\tau\,d\tau=-\pi P(\theta_1+\theta_2)\end{aligned}\right\} \tag{85}$$

Substituting (49) into (85), assuming $\tau=a\cos\varphi$, $\delta=-a\cos\varphi_0$ and fulfilling integration, we will obtain this time, instead of equation (72), equation

$$\left.\begin{aligned} &A_1(\sin\varphi_0-\varphi_0\cos\varphi_0)-A_2(\sin\varphi_0-\varphi_0\cos\varphi_0+\pi\cos\varphi_0)=\frac{\pi\theta}{a},\\ &A_1(\varphi_0-\sin\varphi_0\cos\varphi_0)-A_2(\varphi_0-\sin\varphi_0\cos\varphi_0-\pi)=\frac{\pi P(\theta_1+\theta_2)}{a^2}.\end{aligned}\right\} \tag{86}$$

Substituting (83) into (79), we will find

$$\begin{aligned} M=P\delta+\frac{\theta}{\pi(\theta_1+\theta_2)}\int_{-a}^{a}\sqrt{a^2-\tau^2}\,d\tau-\\ -\frac{1}{\pi(\theta_1+\theta_2)}\int_{-a}^{a}[f_1'(\tau+\delta)+f_2'(\tau+\delta)]\sqrt{a^2-\tau^2}\,d\tau.\end{aligned} \tag{87}$$

Substituting (49) into (87) and producing the same computations as in the derivation of formula (80), we will obtain

$$\begin{aligned} M=-Pa\cos\varphi_0+\frac{2a^3}{\pi(\theta_1+\theta_2)}\left\{\frac{A_1-A_2}{6}\sin^3\varphi_0+\right.\\ \left.+\frac{\cos\varphi_0}{2}[(A_1-A_2)(\varphi_0-\operatorname{tg}\varphi_0)+\pi A_2]\right\}+\frac{\theta a^2}{2(\theta_1+\theta_2)}.\end{aligned} \tag{88}$$

But since compressible bodies can freely revolve about the origin of the coordinates, moment M, which will form pressure $p(x)$ with respect to point $x = 0$, should be equal to zero. According to (88) we obtain the equation

$$\frac{1}{2}(A_+ - A_-)\sin^2\varphi_0 + \cos\varphi_0[(A_+ - A_-)(\varphi_0 - \mathrm{tg}\,\varphi_0) + \pi A_-] = \frac{\pi(\vartheta_1+\vartheta_2)}{a^2} P\cos\varphi_0 - \frac{\pi\theta}{2a}; \tag{89}$$

By adding the first of equations (86) with equation (89), we will find

$$\frac{1}{2}(A_+ - A_-)\sin^2\varphi_0 = \frac{\pi P(\vartheta_1+\vartheta_2)}{a^2}\cos\varphi_0. \tag{90}$$

Dividing (90) by cos ϕ_0 and subtracting from the second of equations (86), we will obtain equation

$$\frac{1}{2}(A_+ - A_-)(2\varphi_0 - 2\sin\varphi_0\cos\varphi_0 - \sin^2\varphi_0\,\mathrm{tg}\,\varphi_0) = -\pi A_-,$$

or

$$\mathrm{tg}\,\varphi_0 + \sin 2\varphi_0 - 2\varphi_0 = \frac{2\pi}{\frac{A_+}{A_-} - 1}. \tag{91}$$

From (90) we find

$$a = \frac{1}{\sin\varphi_0}\sqrt{\frac{2\pi P(\vartheta_1+\vartheta_2)}{(A_+ - A_-)\,\mathrm{tg}\,\varphi_0}}. \tag{92}$$

Multiplying the second of equations (86) by cos ϕ_0 and adding it to the first of these equations, we will find

$$(A_+ - A_-)\sin^2\varphi_0 = \frac{\pi P(\vartheta_1+\vartheta_2)}{a^2}\cos\varphi_0 + \frac{\pi\theta}{2a}. \tag{93}$$

Excluding $A_+ - A_-$ from equations (90) and (93), we will find

$$\theta = \frac{4P(\vartheta_1+\vartheta_2)}{a}\cos\varphi_0. \tag{94}$$

Equation (91) determines the angle ϕ_0, after which formulas (92), (94) and relation $\delta = -a\cos\varphi_0$ determine the half-width of the section of contact a, the displacement of this section relative to the origin of coordinates δ and the relative turn of compressible bodies θ and formulas (69) and (70) determine the pressure $p(x)$ in the region of contact $-a+\delta < x < a+\delta$;

Thus, for the case when the second derivative of the sum of functions $f_1(x)+f_2(x)$ has a jump at point $x = 0$, the contact problem is completely solved by us both when the condition of only one forward displacement of the bodies and according to the condition of forward displacement and their relative turn.

Let us turn now to the case when the second derivative of the sum of functions $f_1(x)+f_2(x)$ turns into infinity at point $x = 0$. We will assume that in the region of contact the indicated sum of functions can be represented in the form

$$f_1(x)+f_2(x)=A|x|^k \qquad (1<k<2). \tag{95}$$

Substituting (95) into (14), we will have

$$\left.\begin{aligned} f(x)&=\frac{\varepsilon-A|x|^k}{\vartheta_1+\vartheta_2},\\ f'(x)&=-\frac{Ak|x|^{k-1}}{\vartheta_1+\vartheta_2} \quad \text{when } x>0,\\ f'(x)&=\frac{Ak|x|^{k-1}}{\vartheta_1+\vartheta_2} \quad \text{when } x<0. \end{aligned}\right\} \tag{96}$$

Substituting (96) into formula (34) for pressure $p(x)$, we will find

$$p(x)=\frac{Ak\sqrt{a^2-x^2}}{\pi^2(\vartheta_1+\vartheta_2)}\left[-\int_{-a}^{0}\frac{|t|^{k-1}dt}{\sqrt{a^2-t^2}(t-x)}+\int_{0}^{a}\frac{|t|^{k-1}dt}{\sqrt{a^2-t^2}(t-x)}\right]. \tag{97}$$

Replacing t by $-t$, we will obtain

$$-\int_{-a}^{0}\frac{|t|^{k-1}dt}{\sqrt{a^2-t^2}(t-x)}=\int_{0}^{a}\frac{|t|^{k-1}dt}{\sqrt{a^2-t^2}(t+x)}. \tag{98}$$

Substituting (98) into (97), we will find

$$p(x)=\frac{2Ak\sqrt{a^2-x^2}}{\pi^2(\vartheta_1+\vartheta_2)}\int_{0}^{a}\frac{t^k\,dt}{\sqrt{a^2-t^2}(t^2-x^2)}. \tag{99}$$

Condition (32) in this case is fulfilled, since function $f'(x)$, determined by relationship (96), is odd. Substituting (96) into (33), we will find

$$\int_{-a}^{0} \frac{|t|^k \, dt}{\sqrt{a^2-t^2}} + \int_{0}^{a} \frac{|t|^k \, dt}{\sqrt{a^2-t^2}} = \frac{\pi P(\theta_1+\theta_2)}{Ak},$$

or

$$2\int_{0}^{a} \frac{t^k \, dt}{\sqrt{a^2-t^2}} = \frac{\pi P(\theta_1+\theta_2)}{Ak}. \qquad (100)$$

Assuming in (99) and (100) $t = a\tau$, we will find

$$p(x) = \frac{2Ak a^{k-1}}{\pi^2(\theta_1+\theta_2)} \sqrt{1-\frac{x^2}{a^2}} \int_{0}^{1} \frac{\tau^k \, d\tau}{\sqrt{1-\tau^2}\left(\tau^2-\frac{x^2}{a^2}\right)}, \qquad (101)$$

$$a^k \int_{0}^{1} \frac{\tau^k \, d\tau}{\sqrt{1-\tau^2}} = \frac{\pi P(\theta_1+\theta_2)}{2Ak}. \qquad (102)$$

Substituting A from (102) into (101), we will obtain formula

$$p(x) = \frac{P}{\pi a} \, \frac{\sqrt{1-\frac{x^2}{a^2}} \displaystyle\int_{0}^{1} \frac{\tau^k \, d\tau}{\sqrt{1-\tau^2}\left(\tau^2-\frac{x^2}{a^2}\right)}}{\displaystyle\int_{0}^{1} \frac{\tau^k \, d\tau}{\sqrt{1-\tau^2}}}. \qquad (103)$$

From (102) we will find

$$a = \left[\frac{\pi P(\theta_1+\theta_2)}{2Ak \displaystyle\int_{0}^{1} \frac{\tau^k \, d\tau}{\sqrt{1-\tau^2}}} \right]^{\frac{1}{k}}. \qquad (104)$$

Formula (104) determines the half-width of the section of contact a, and formula (103) – pressure $p(x)$. Definite integrals entering into formulas (103) and (104), when $1 < k < 2$, are not expressed in terms of elementary functions. When $k = 3/2$ these definite integrals (elliptic), after reduction to canonical form, can be calculated from tables[1] available for elliptic integrals. Figure 11 shows the graph

[1]See Appendix 1, p. 6 .

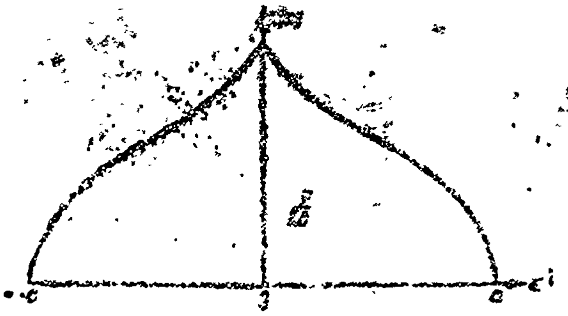

Fig. 11.

of pressure $p(x)$ calculated from formula (103) for the case $k = 3/2$. Pressure $p(x)$ is limited in the whole region of contact $-a \leq x \leq a$, however, derivative $p'(x)$ undergoes discontinuity at point $x = 0$, and the curve depicting function $p(x)$ has when $x = 0$ a corner point.

In conclusion of this paragraph, let us note the maximum case of the examined problem when $k = 1$, according to the formula (96)

$$f(x) = \frac{\varepsilon - A|x|}{\vartheta_1 + \vartheta_2}. \qquad (105)$$

In Chapter I we showed that for the case when the right side of integral equation (17) $f(x)$ has the form

$$f(x) = \varepsilon - A|x|, \qquad (106)$$

this equation has the solution

$$p(x) = -\frac{2A}{\pi^2} \ln \frac{a - \sqrt{a^2 - x^2}}{|x|}, \qquad (107)$$

which becomes zero when $|x| = a$, if

$$\varepsilon = \frac{2Aa}{\pi} \qquad (108)$$

(formulas (203) and (204) of Chapter I).

Replacing in (107) and (108) A by $\frac{A}{\vartheta_1 + \vartheta_2}$, we will obtain the solution to the problem for the case when function $f(x)$ has the form (105). We find

$$p(x) = -\frac{2A}{\pi(\theta_1+\theta_2)} \ln \frac{a - \sqrt{a^2 - x^2}}{|x|} \qquad (109)$$

when

$$P = \frac{2Aa}{\pi(\theta_1+\theta_2)}. \qquad (110)$$

Substituting A from (110) into (109), we will obtain formula

$$p(x) = -\frac{P}{\pi a} \ln \frac{a - \sqrt{a^2 - x^2}}{|x|}. \qquad (111)$$

From (110) we Find

$$a = \frac{\pi P(\theta_1+\theta_2)}{2A}. \qquad (112)$$

Formula (112) determines the half-width of the section of contact a, and formula (111) – pressure $p(x)$. Figure 12a shows the graph of function $p(x)$ determined by formula (111). At point $x = 0$ the pressure $p(x)$ becomes infinity. As can be seen from (95), for the examined case ($k = 1$) the opening between compressible bodies prior to compression is determined by formula

$$f_1(x) + f_2(x) = A|x|, \qquad (113)$$

and the configuration of compressible bodies in the neighborhood of the point of their initial contact has the form shown in Fig. 12b. Thus, the examined maximum case $k = 1$ corr sponds to the compression of two wedges or the pressure of a wedge on the rectilinear boundary of the elastic medium. In this maximum case at point $x = 0$ it is not the second derivative of the sum of functions $f_1(x) + f_2(x)$, that undergoes discontinuity but the first derivative:

$$\left.\begin{aligned} f_1'(x) + f_2'(x) &= -A \quad \text{when } x = -0, \\ f_1'(x) + f_2'(x) &= A \quad \text{when } x = +0. \end{aligned}\right\} \qquad (114)$$

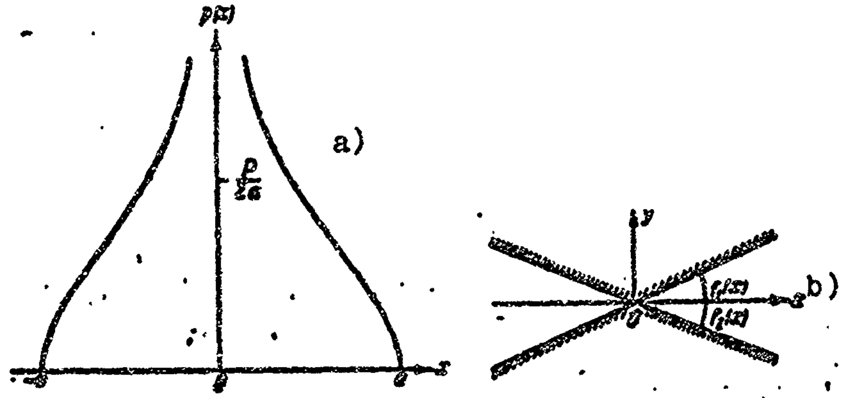

Fig. 12.

§ 3. The Problem of Pressure of a Rigid Stamp on an Elastic Half-Plane

In the preceding section we examined in detail the case of the contact problem in which the initial contact of compressible bodies in plane xOy is carried out at one point. Let us now consider the case when the initial contact of compressible bodies in plane xOy is carried out not at the point but along a certain segment of the Ox axis. If the length of this segment is designated by $2a$ and the origin of the coordinates is located in the middle of this segment, then the set of points S_0, in which the initial contact between the compressible bodies is carried out, will constitute the segment of the Ox axis $-a<x<a$.

Let us examine first the case when one of the compressible bodies has the form of a stamp with right angles in the section by plane xOy. Usually in this problem this body can be considered as rigid, and the problem is formulated as a problem about pressure of the rigid stamp on an elastic half-plane. In this case and after compression the contact between the compressible bodies will be carried out along the segment of the Ox axis $-a < x < a$, and according to general formulas (13) and (14) the pressure $p(x)$ under the stamp will be determined by the integral equation

$$\int_{-a}^{a} p(t)\ln\frac{1}{|t-x|}\,dt = \epsilon, \quad -a<x<a, \tag{115}$$

where a — certain constant, since when $-a < x < a$ the initial opening between the compressible bodies $f_1(x)+f_2(x)=0$. In Chapter I we showed that equation (115) has the solution

$$p(x)=\frac{P}{\pi\sqrt{a^2-x^2}}, \tag{116}$$

where $P=\int_{-a}^{a} p(x)\,dx$ — the compressing force (see formula (74) of Chapter I). Figure 13 shows a graph of pressure $p(x)$ under a stamp plotted in accordance with formula (116). Pressure $p(x)$ increases without limit with the approach to boundaries of the section of contact $x = -a$ and $x = +a$. At the end of the preceding chapter we already encountered the case of conversion into infinity of pressure $p(x)$ when we examined the pressure of the wedge on an elastic half-plane. In reality the real profile of the elastic body will never have corner points, so that the wedge or stamp, which have right angles in the section, are abstractions, which lead with solution of the contact problem to an unreal distribution of pressure in the region of contact. Below we examine the problem about the pressure of a stamp on an elastic half-plane, considering that the profile of the stamp has a continuously revolving tangent. Thus, if

$$y=f_1(x)$$

is the equation of the curve limiting the stamp (Fig. 14) in the section by plane xOy, we will, as earlier, consider that

$$f_1(x)=0 \text{ when } -a\leqslant x\leqslant a, \tag{117}$$

and in the neighborhood of points $x = -a$ and $x = a$ when $|x|>a$ we will approximately represent function $f_1(x)$ by the first term of its expansion in Taylor series:

$$\left.\begin{aligned} f_1(x)&=\frac{1}{2}f_1''(a+0)\,(x-a)^2 \quad \text{when } x>a,\\ f_1(x)&=\frac{1}{2}f_1''(-a-0)\,(a+x)^2 \quad \text{when } x<-a, \end{aligned}\right\} \tag{118}$$

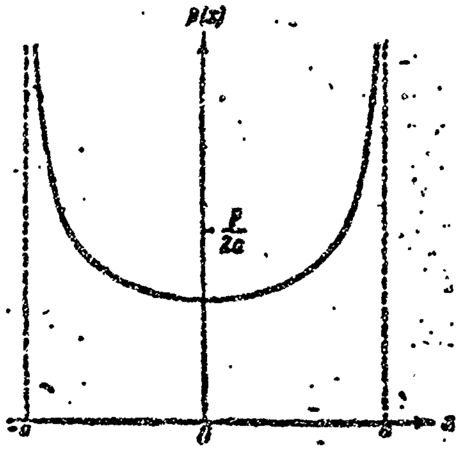

Fig. 13.

Fig. 14.

considering that with the approach to point $x = a$ on the right and to point $x = -a$ on the left the second derivative $f_1''(x)$ approaches finite values different from zero. These values will be considered equal, assuming the stamp to be symmetric, and we will designate by A

$$f_1''(a+0) = f_1''(-a-0) = A. \qquad (119)$$

For simplicity considering that the elastic body on which the stamp presses has a rectilinear limit, coinciding with the Ox axis, let us take

$$f_2(x) = 0. \qquad (120)$$

Substituting (117), (118) and (120) into (14) and taking into account notation (119), we will find

$$\left.\begin{aligned} f(x) &= -\frac{A}{2(\vartheta_1+\vartheta_2)}(a+x)^2 + \text{const} \quad \text{when} \quad x < -a, \\ f(x) &= \text{const.} \quad \text{when} \quad -a < x < a, \\ f(x) &= -\frac{A}{2(\vartheta_1+\vartheta_2)}(x-a)^2 + \text{const} \quad \text{when} \quad x > a, \end{aligned}\right\} \qquad (121)$$

whence

$$\left.\begin{aligned} f'(x) &= -\frac{A}{\vartheta_1+\vartheta_2}(a+x) \text{ when } x \leqslant -a, \\ f'(x) &= 0 \text{ when } -a \leqslant x \leqslant a, \\ f'(x) &= -\frac{A}{\vartheta_1+\vartheta_2}(x-a) \text{ when } x \geqslant a. \end{aligned}\right\} \tag{122}$$

Designating by $2b$ the width of the region of contact after compression ($b > a$), we will find the pressure $p(x)$ according to the general formula (34), replacing in it the half-width of the section of contact a by b:

$$p(x) = -\frac{1}{\pi^2}\sqrt{b^2-x^2}\int_{-b}^{b}\frac{f'(t)\,dt}{\sqrt{b^2-t^2}\,(t-x)}. \tag{123}$$

Substituting (122) into (123), we will find

$$p(x) = \frac{A\sqrt{b^2-x^2}}{\pi^2(\vartheta_1+\vartheta_2)}\left[\int_{-b}^{-a}\frac{(a+t)\,dt}{\sqrt{b^2-t^2}\,(t-x)} + \int_{a}^{b}\frac{(t-a)\,dt}{\sqrt{b^2-t^2}\,(t-x)}\right],$$

or

$$p(x) = \frac{A\sqrt{b^2-x^2}}{\pi^2(\vartheta_1+\vartheta_2)}\left[(a+x)\int_{-b}^{-a}\frac{dt}{\sqrt{b^2-t^2}\,(t-x)} + \int_{-b}^{-a}\frac{dt}{\sqrt{b^2-t^2}} + \right.$$
$$\left. + \int_{a}^{b}\frac{dt}{\sqrt{b^2-t^2}} + (x-a)\int_{a}^{b}\frac{dt}{\sqrt{b^2-t^2}\,(t-x)}\right]. \tag{124}$$

Let us find

$$\int_{-b}^{-a}\frac{dt}{\sqrt{b^2-t^2}} = \int_{a}^{b}\frac{dt}{\sqrt{b^2-t^2}} = \frac{\pi}{2} - \varphi_0, \tag{125}$$

where

$$\varphi_0 = \arcsin\frac{a}{b}. \tag{126}$$

Assuming

$$t = b\,\frac{2\tau}{1+\tau^2}, \quad x = b\,\frac{2\xi}{1+\xi^2}, \tag{127}$$

we will find

$$\int_{x_1}^{x_2} \frac{dx}{\sqrt{b^2-x^2}\,(s-x)} = \frac{1+\xi^2}{b} \int_{\xi_1}^{\xi_2} \frac{d\tau}{(\tau-\xi)(1-\tau\xi)} =$$

$$= \frac{1+\xi^2}{b(1-\xi^2)} \left(\int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\xi} - \int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\frac{1}{\xi}} \right), \tag{128}$$

where ξ_1 and ξ_2 are connected with x_1 and x_2 by relation

$$x_1 = b\frac{2\xi_1}{1+\xi_1^2}, \quad x_2 = b\frac{2\xi_2}{1+\xi_2^2}. \tag{129}$$

Let us find further:

when $\xi < \xi_1$

$$\int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\xi} = \ln(\tau-\xi)\Big|_{\tau=\xi_1}^{\tau=\xi_2} = \ln\frac{\xi_2-\xi}{\xi_1-\xi}, \tag{130}$$

when $\xi > \xi_2$

$$\int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\xi} = \ln(\xi-\tau)\Big|_{\tau=\xi_1}^{\tau=\xi_2} = \ln\frac{\xi-\xi_2}{\xi-\xi_1}, \tag{131}$$

when $\xi_1 < \xi < \xi_2$

$$\int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\xi} = \lim_{\varepsilon\to 0}\left(\int_{\xi_1}^{\xi-\varepsilon} \frac{d\tau}{\tau-\xi} + \int_{\xi+\varepsilon}^{\xi_2} \frac{d\tau}{\tau-\xi} \right) =$$

$$= \lim_{\varepsilon\to 0}\left[\ln(\xi-\tau)\Big|_{\tau=\xi_1}^{\tau=\xi-\varepsilon} + \ln(\tau-\xi)\Big|_{\tau=\xi+\varepsilon}^{\tau=\xi_2} \right] =$$

$$= \lim_{\varepsilon\to 0} \ln\frac{\varepsilon(\xi_2-\xi)}{(\xi-\xi_1)\varepsilon} = \ln\frac{\xi_2-\xi}{\xi-\xi_1}. \tag{132}$$

Formulas (130), (131) and (132) can be united into one:

$$\int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\xi} = \ln\left|\frac{\xi-\xi_2}{\xi-\xi_1}\right|. \tag{133}$$

Analogously,

$$\int_{\xi_1}^{\xi_2} \frac{d\tau}{\tau-\frac{1}{\xi}} = \ln\left|\frac{1-\xi\xi_2}{1-\xi\xi_1}\right|. \tag{134}$$

Substituting (133) and (134) into (128), we will find

$$\int_{x_1}^{x_2} \frac{dt}{\sqrt{b^2-t^2}\,(t-x)} = \frac{1+\xi^2}{b(1-\xi^2)} \ln \left| \frac{(\xi-\xi_2)(1-\xi\xi_1)}{(\xi-\xi_1)(1-\xi\xi_2)} \right|. \tag{135}$$

Taking into account (126), we will find from (129):

$$\left.\begin{aligned} &\text{when } x_1 = -b, \quad x_2 = -a, \; \xi_1 = -1, \; \xi_2 = -\operatorname{tg}\frac{\varphi_0}{2}, \\ &\text{when } x_1 = a, \qquad x_2 = b, \quad \xi_1 = \operatorname{tg}\frac{\varphi_0}{2}, \; \xi_2 = 1. \end{aligned}\right\} \tag{136}$$

Assuming further

$$\xi = \operatorname{tg}\frac{\varphi}{2} \tag{137}$$

and substituting (136) and (137) into (135), we will find

$$\left.\begin{aligned} &\int_{-b}^{-a} \frac{dt}{\sqrt{b^2-t^2}\,(t-x)} = \frac{1}{b\cos\varphi} \ln \left| \frac{\sin\frac{\varphi+\varphi_0}{2}}{\cos\frac{\varphi-\varphi_0}{2}} \right|, \\ &\int_{a}^{b} \frac{dt}{\sqrt{b^2-t^2}\,(t-x)} = \frac{1}{b\cos\varphi} \ln \left| \frac{\cos\frac{\varphi+\varphi_0}{2}}{\sin\frac{\varphi-\varphi_0}{2}} \right|. \end{aligned}\right\} \tag{138}$$

From (127) and (137) it follows that

$$x = b\sin\varphi. \tag{139}$$

Substituting into (124), (125), (138) and (139), and assuming according to (126),

$$a = b\sin\varphi_0, \tag{140}$$

we will find

$$p = \frac{Ab\cos\varphi}{\pi^2(\theta_1+\theta_2)} \left(\frac{\sin\varphi+\sin\varphi_0}{\cos\varphi} \ln \left| \frac{\sin\frac{\varphi+\varphi_0}{2}}{\cos\frac{\varphi-\varphi_0}{2}} \right| + \right.$$

$$\left. + \frac{\sin\varphi-\sin\varphi_0}{\cos\varphi} \ln \left| \frac{\cos\frac{\varphi+\varphi_0}{2}}{\sin\frac{\varphi-\varphi_0}{2}} \right| + \pi - 2\varphi_0 \right);$$

or

$$p=\frac{Ab}{\pi^2(\theta_1+\theta_2)}\Big[(\pi-2\varphi_0)\cos\varphi+\sin\varphi\ln\left|\frac{\sin(\varphi+\varphi_0)}{\sin(\varphi-\varphi_0)}\right|+$$
$$+\sin\varphi_0\ln\left|\operatorname{tg}\frac{\varphi+\varphi_0}{2}\operatorname{tg}\frac{\varphi-\varphi_0}{2}\right|\Big]. \tag{141}$$

Substituting (122) into condition (33)[1] we will obtain equation

$$\int_{-b}^{-a}\frac{(a+t)\,t\,dt}{\sqrt{b^2-t^2}}+\int_{a}^{b}\frac{(t-a)\,t\,dt}{\sqrt{b^2-t^2}}=\frac{\pi P(\theta_1+\theta_2)}{A}. \tag{142}$$

Assuming in (142) $t = b \sin \phi$ and $a = b \sin \phi_0$, according to (140), we will find

$$\int_{-\frac{\pi}{2}}^{-\varphi_0}(\sin\varphi_0+\sin\varphi)\sin\varphi\,d\varphi+\int_{\varphi_0}^{\frac{\pi}{2}}(\sin\varphi-\sin\varphi_0)\sin\varphi\,d\varphi=\frac{\pi P(\theta_1+\theta_2)}{Ab^2},$$

or, fulfilling integration

$$\frac{\pi}{2}-\varphi_0-\frac{\sin 2\varphi_0}{2}=\frac{\pi P(\theta_1+\theta_2)}{Ab^2}. \tag{143}$$

Substituting b from (140) into (139), (141) and (143), we will find

$$\frac{\pi-2\varphi_0}{2\sin^2\varphi_0}-\operatorname{ctg}\varphi_0=\frac{\pi P(\theta_1+\theta_2)}{Aa^2}, \tag{144}$$

$$x=a\frac{\sin\varphi}{\sin\varphi_0}, \tag{145}$$

$$p=\frac{Aa}{\pi^2(\theta_1+\theta_2)\sin\varphi_0}\Big[(\pi-2\varphi_0)\cos\varphi+\sin\varphi\ln\left|\frac{\sin(\varphi+\varphi_0)}{\sin(\varphi-\varphi_0)}\right|+$$
$$+\sin\varphi_0\ln\left|\operatorname{tg}\frac{\varphi+\varphi_0}{2}\operatorname{tg}\frac{\varphi-\varphi_0}{2}\right|\Big]. \tag{146}$$

Substituting A from (144) into (146), we will find

[1] In accordance with designation b, accepted in the examined problem for the half-width of the section of contact, it is necessary here in (33) to replace a by b.

$$p = \frac{2P\sin\varphi_0\left[(\pi-2\varphi_0)\cos\varphi + \sin\varphi\ln\left|\frac{\sin(\varphi+\varphi_0)}{\sin(\varphi-\varphi_0)}\right|\right]}{\pi a(\pi-2\varphi_0-\sin 2\varphi_0)} + \frac{\sin^2\varphi_0\ln\left|\operatorname{tg}\frac{\varphi+\varphi_0}{2}\operatorname{tg}\frac{\varphi-\varphi_0}{2}\right|}{\pi a(\pi-2\varphi_0-\sin 2\varphi_0)}, \quad -\frac{\pi}{2}\leqslant\varphi\leqslant\frac{\pi}{2}. \tag{147}$$

Formula (144) determines the angle ϕ_0, after which formulas (145) and (147) determine the pressure $p(x)$ in the region $-\frac{\pi}{2}<\varphi<\frac{\pi}{2}$, which corresponds to the region of contact $-b<x<b$ $\left(b=\frac{a}{\sin\varphi_0}\right)$. Figure 15 shows graphs of pressure $p(x)$ corresponding to different values of angle ϕ_0, i.e., different ratios $k=\frac{b}{a}\left(\frac{b}{a}=\frac{1}{\sin\varphi_0}\right)$. Graphs of pressure $p(x)$ shown in Fig. 15 correspond to the identical half-width of the base of the stamp a (see Fig. 14), identical compressing force P and identical elastic constants, but different values of A, i.e., different curvature of edges of the stamp, owing to which angle ϕ_0 (see formula (144)) and half-width of the section of contact $b=\frac{a}{\sin\varphi_0}$ changes.

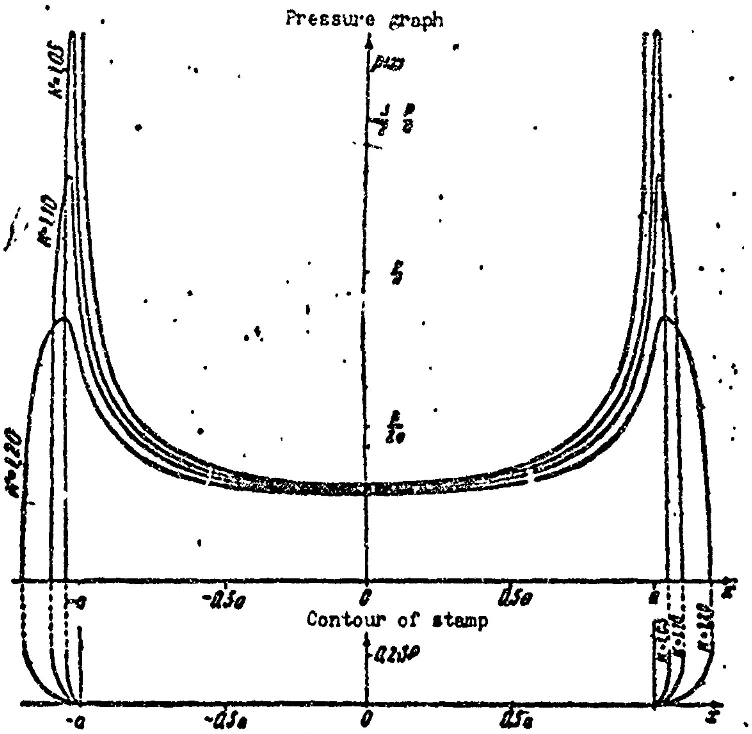

Fig. 15.

Thus, considering the curvature of edges of the stamp to be limited and assuming thus the real conditions as a basis of the problem about the pressure of the stamp, we arrive at the real picture of distribution of pressure under the stamp. As can be seen from Fig. 15, an increase in curvature of edges of the stamp, other things being equal, involves an increase in maximum pressure under the stamp; however, this pressure remains limited, and as yet the curvature of the stamp is limited.

Formulas (144), (145) and (147) given by us make it possible to set the maximum pressure under the stamp, if the configuration of the stamp (width of its base $2a$ and curvature of its edges, i.e., A), elastic constants and compressing force P are assigned. The graph shown in Fig. 16 makes it possible immediately to detect from these data the width of the area element of contact b, not solving equations (144), after which by formulas (145) and (147) it is possible to calculate the pressure $p(x)$ at any point of contact.

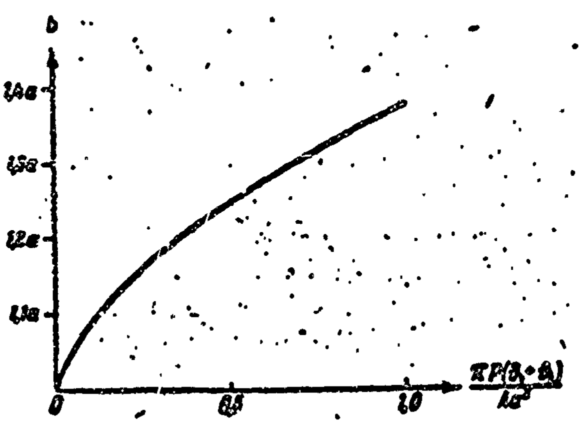

Fig. 16.

§ 4. Case of Several Sections of Compression

Thus far we assumed that after compression the contact of the compressed elastic bodies is carried out along one segment of the Ox axis, i.e., the set of points S in which there occurs contact of compressed bodies constitutes one continuous line. Let us consider now the case when the contact of the compressed elastic bodies is carried out along several segments of the Ox axis: (a_1, b_1),

$(a_2, b_2), \ldots, (a_n, b_n)$ (Fig. 17). In this case the basic integral equation of the contact problem (13) will have the form

$$\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{|t-x|}\,dt = f(x), \quad a_m < x < b_m \ (m = 1, 2, \ldots, n), \tag{148}$$

where as before

$$f(x) = \frac{c - f_1(x) - f_2(x)}{\theta_1 + \theta_2}. \tag{149}$$

The integral equation (148) completely coincides with equation (209) of Chapter I. As we showed in Chapter I, the solution of this equation has the form

$$p(x) = \frac{(-1)^{n-l+1}\left[\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x} + P_{n-1}(x)\right]}{\pi\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}, \quad a_l < x < b_l, \tag{150}$$

where $P_{n-1}(x)$ — polynomial of the power $n - 1$:

$$P_{n-1}(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_{n-2}x^{n-2} - Px^{n-1}, \tag{151}$$

P — compressing force, i.e.,

$$P = \sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt \tag{152}$$

(see formulas (244), (245) and (246) of Chapter I).

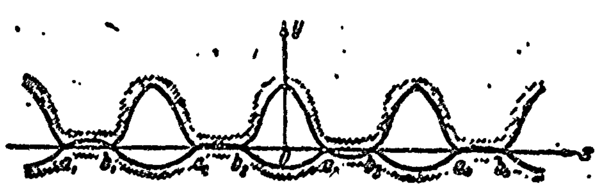

Fig. 17.

Coefficients c_0, c_1, c_2, ..., c_{n-2} of polynomial $P_{n-1}(x)$ are determined from the system of linear equations:

$$\sum_{i=0}^{n-2} c_i \int_{b_m}^{a_{m+1}} \frac{x^i\,dx}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} =$$

$$= (-1)^{n-m}\left[f(a_{m+1}) - f(b_m)\right] + P\int_{b_m}^{a_{m+1}} \frac{x^{n-1}\,dx}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}} -$$

$$-\frac{1}{\pi}\int_{b_m}^{a_{m+1}} \frac{1}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}\times$$

$$\times\left[\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x}\right]dx, \qquad (153)$$

$$m = 1, 2, \ldots, n-1,$$

(see formula (252) of Chapter I).

Boundaries of sections of contact, i.e., abscissas a_1, b_1; a_2, b_2, ..., a_n, b_n, will be determined in general from the condition that the pressure $p(x)$ should remain limited in the whole region of contact, including the boundaries of sections of contact $x = a_1$, $x = b_1$, ..., $x = a_n$, $x = b_n$. As can be seen from formula (150), this is possible only when the numerator in this formula becomes zero when $x = a_l$, $x = b_l$ ($l = 1, 2, \ldots, n$). Hence we get $2n$ equations

$$\left.\begin{aligned}
&\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\times\\
&\qquad\qquad\times\frac{f'(t)\,dt}{t-a_l} + P_{n-1}(a_l) = 0,\\
&\frac{1}{\pi}\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\times\\
&\qquad\qquad\times\frac{f'(t)\,dt}{t-b_l} + P_{n-1}(b_l) = 0,\\
&\qquad l = 1, 2, \ldots, n,
\end{aligned}\right\} \qquad (154)$$

which determine abscissas $a_1, b_1, a_2, b_2, \ldots, a_n, b_n$.

The formulas obtained by us completely solve the examined problem by determining the region of contact of compressed elastic bodies and pressure $p(x)$ in the region of contact. In the special case when some of the boundaries of the sections of contact are predetermined beforehand by the configuration of the compressible bodies, as in the case of the pressure of the stamp with right angles on the elastic half-plane, examined in the preceding paragraph, the appropriate equations of (154) must be dropped.

Let us examine the simplest example in which the stamp, having the configuration shown in Fig. 18, presses on the elastic half-plane. We will assume clearance δ to be so small that with compression the contact of the stamp with the elastic half-plane is carried out along two sections of the Ox axis: $-b<x<-a$ and $a<x<b$. In this case

$$n=2,\ a_1=-b,\ b_1=-a,\ a_2=-a,\ b_2=b,$$

$$f_1(x)=0 \text{ when } -b<x<-a,\ f_1(x)=\delta \text{ when } a<x<b, \tag{155}$$

$$f_2(x)=0, \tag{156}$$

and according to (149)

$$f(x)=\alpha \text{ when } -b<x<-a,\ f(x)=\alpha+\varepsilon \text{ when } a<x<b, \tag{157}$$

where

$$\varepsilon=-\frac{\delta}{\vartheta_1+\vartheta_2} \tag{158}$$

(if one were to consider the stamp to be absolutely rigid, then $\vartheta_1=0$), α – certain indefinite constant.

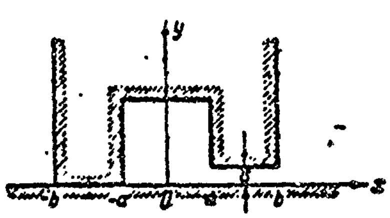

Fig. 18.

Solution of the integral equation (148) according to conditions (155) and (157) was obtained by us in Chapter I and according to the formulas (256) and (261) of Chapter I has the form

$$p(x) = \pm \frac{c_0 - Px}{\pi \sqrt{(x^2 - a^2)(b^2 - x^2)}},$$
$$a < |x| < b, \tag{159}$$

where the plus sign prevails when $x < 0$ and minus sign when $x > 0$,

$$c_0 = -\frac{b\varepsilon}{2K(k)},$$

or, according to (158),

$$c_0 = \frac{b\delta}{2(\vartheta_1 + \vartheta_2)K(k)}, \tag{160}$$

where $K(k)$ - complete elliptic integral of the first kind:

$$K(k) = \int_0^1 \frac{dx}{\sqrt{(1 - x^2)(1 - k^2x^2)}},$$

the modulus of which k is equal to

$$k = \frac{a}{b}. \tag{161}$$

Figure 19 shows the graph of pressure $p(x)$ for the case

$$k = \frac{a}{b} = 0{,}4, \quad \delta = 0{,}656(\vartheta_1 + \vartheta_2)P.$$

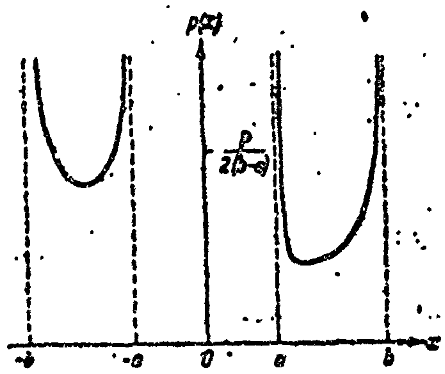

Fig. 19.

On boundaries of sections of contact the pressure $p(x)$ turns into infinity, which is the corollary of the condition accepted by us about the presence of right angles on the profile of the stamp. General formulas given in this chapter make it possible to solve the contact problem in the case when we consider the curvature of the edges of the stamp limited.

In this case we will obtain the real picture of the distribution of pressure under the stamp, at which the pressure will be limited everywhere in the region of contact. In view of cumbersomeness of computations we will not discuss this in detail.

Let us now examine the case of the contact problem when the elastic body, located in the lower half-plane, is pressed by several separate bodies lying in the upper half-plane, and forces P_1, P_2, ..., P_n, pressing these bodies are assigned.

If in this case by function $f_1(x)$ we understand the function determining in various intervals of the argument x the configuration of each of the bodies lying in the upper half-plane, then the contact problem will be solved by the same integral equation (148), the only difference being that constant c, which enters into formula (149), can have different values on each of the sections of contact, since each of the compressible bodies can accomplish according to compression its own forward displacement.

As we showed in Chapter I, pressure $p(x)$, which is the solution to equation (148), will in this case be determined by the same formula (150) with the only difference being that coefficients of the polynomial $P_{n-1}(x) - c_0, c_1, \ldots, c_{n-2}$ will this time be determined no longer by equations (153) but by equations (255) of Chapter I:

$$\sum_{l=0}^{n-2} c_l \int_{a_m}^{b_m} \frac{x^l\,dx}{\sqrt{\left|\prod_{m=1}^{n} (x-a_m)(x-b_m)\right|}} = (-1)^{n-m+1}\pi P_m + P \int_{a_m}^{b_m} \frac{x^{n-1}\,dx}{\sqrt{\left|\prod_{m=1}^{n} (x-a_m)(x-b_m)\right|}}$$

$$-\frac{1}{\pi}\int_{a_m}^{b_m}\frac{1}{\sqrt{\left|\prod_{m=1}^{n}(x-a_m)(x-b_m)\right|}}\times$$

$$\times\left[\sum_{m=1}^{n}(-1)^{n-m}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}(t-a_m)(t-b_m)\right|}\frac{f'(t)\,dt}{t-x}\right]dx, \tag{162}$$

$$m=1,\ 2,\ \ldots,\ n-1,$$

where

$$P=P_1+P_2+\ldots+P_n. \tag{163}$$

Let us examine the simplest example in which the elastic half-plane is pressed by two stamps pressed by forces P_1 and P_2 (Fig. 20). In this case $n = 2$, $a_1 = -b$, $b_1 = -a$, $a_2 = a$, $b_2 = b$,

$$f(x)=0 \text{ when } -b<x<-a \text{ and } a<x<b. \tag{164}$$

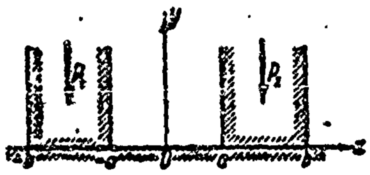

Fig. 20.

As we showed in Chapter I, the integral equation (148) according to conditions (164) has the solution

$$p(x)=\pm\frac{\frac{\pi b}{2K(k)}(P_1-P_2)-(P_1+P_2)x}{\pi\sqrt{(x^2-a^2)(b^2-x^2)}}, \tag{165}$$

$$k=\sqrt{1-\frac{a^2}{b^2}},\quad a<|x|<b$$

(see formula (267) of Chapter I), where the plus sign prevails when $x < 0$, and minus sign when $x > 0$, and $K(k)$ is the complete elliptic integral of the first kind with modulus k,

$P_1 = \int_{-b}^{-a} p(x)\,dx$ and $P_2 = \int_{a}^{b} p(x)\,dx$ — compressing forces.

Formulas (159) and (165) differ only by the constant component in the numerator of the fraction determining the pressure $p(x)$. When $P_1 > P_2$ the distribution of pressure under the stamps will have the same character as that on Fig. 15.

§ 5. Periodic Contact Problem

Let us examine now the case of the contact problem when functions $f_1(x)$ and $f_2(x)$, which determine the configuration of compressible bodies, are periodic with period l (Fig. 21a) so that the number of sections of the contact is infinitely great. Let us denote by $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$,

$$0 < a_1 < b_1 < a_2 < b_2 < \ldots < a_n < b_n < l, \tag{166}$$

sections of contact of compressed bodies in the interval $0 < x < l$, i.e., within limits of one period (we will consider that the origin of the coordinates lies at the point where the contact between the compressed bodies is absent). Pressure $p(x)$ in the region of contact will be a periodic function with period l. Let us consider the action of the periodic normal pressure $p(x)$ on the lower elastic half-plane (Fig. 21b).

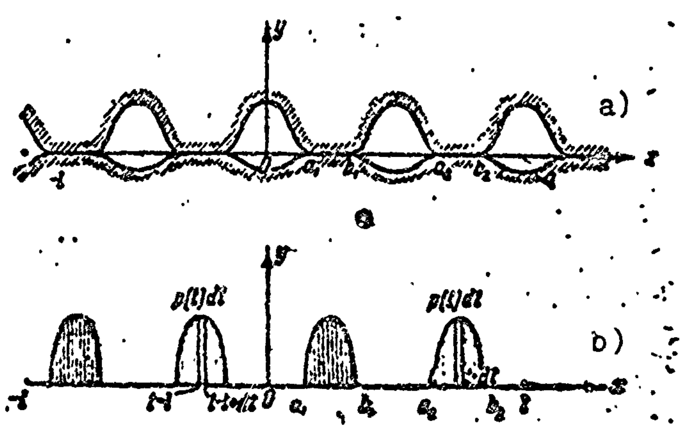

Fig. 21.

Let us assume that $x = t$ is some point of the interval $0 < x < l$, in which on the boundary of the elastic medium pressure $p(t)$ acts. Then acting on the interval $(t, t+dt)$ will be the elementary force $p(t)dt$, which in the end point of the elastic medium with abscissa x will cause displacement dv in the direction of the Oy axis equal to

$$dv = -\theta p(t) \ln \frac{1}{|t-x|} dt + \text{const.},$$

according to the formula (4).

Since function $p(x)$ is periodic, at points $x = t + nl$ ($n = \ldots -2, -1, 1, 2, \ldots$) pressure $p(t)$ will also act on the boundary of the elastic medium. Acting on interval $(t+nl, t+nl+dt)$ will be the elementary force $p(t)dt$, which in the end point of the elastic medium with abscissa x will cause the elementary displacement:

$$dv = -\theta p(t) \ln \frac{1}{|t+nl-x|} dt + \text{const.} = \theta p(t) \ln \left|1 + \frac{t-x}{nl}\right| dt + \text{function } (t)$$

Having summed up the elementary displacements, which appear due to the pressure acting on intervals

$$\ldots, (t-2l, t-2l+dt), \quad (t-l, t-l+dt), \quad (t, t+dt),$$
$$(t+l, t+l+dt), \quad (t+2l, t+2l+dt),$$

we will obtain the elementary displacement:

$$dv = \theta p(t) \left(\ldots + \ln\left|1 - \frac{t-x}{2l}\right| + \ln\left|1 - \frac{t-x}{l}\right| + \ln|t-x| + \right.$$
$$\left. + \ln\left|1 + \frac{t-x}{l}\right| + \ln\left|1 + \frac{t-x}{2l}\right| + \ldots \right) = \theta p(t) \ln \left|(t-x)\left[1 - \right.\right.$$
$$\left.\left. - \frac{(t-x)^2}{l^2}\right]\left[1 - \frac{(t-x)^2}{4l^2}\right]\left[1 - \frac{(t-x)^2}{9l^2}\right] \ldots \right| dt + \text{function } (t) \qquad (167)$$

As is known, function $\sin u$ can be represented by the infinite product

$$\sin u = u\left(1 - \frac{u^2}{\pi^2}\right)\left(1 - \frac{u^2}{4\pi^2}\right)\left(1 - \frac{u^2}{9\pi^2}\right) \ldots \qquad (168)$$

$$(\vartheta_1+\vartheta_2)\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{2\left|\sin\frac{\pi(t-x)}{l}\right|}\,dt = c - f_1(x) - f_2(x), \tag{173}$$

$$a_m < x < b_m \ (m = 1, 2, \ldots, n),$$

where c is a certain constant, or

$$\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\ln\frac{1}{2\left|\sin\frac{\pi(t-x)}{l}\right|}\,dt = f(x), \tag{174}$$

and $a_m < x < b_m \ (m = 1, 2, \ldots, n)$, where

$$f(x) = \frac{c - f_1(x) - f_2(x)}{\vartheta_1+\vartheta_2}. \tag{175}$$

Assuming in (174)

$$t = \frac{l}{2\pi}\varphi, \quad x = \frac{l}{2\pi}\theta, \tag{176}$$

we will obtain equation

$$\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m} p\left(\frac{l\varphi}{2\pi}\right)\ln\frac{1}{2\left|\sin\frac{\varphi-\theta}{2}\right|}\,d\varphi = \frac{2\pi}{l} f\left(\frac{l\theta}{2\pi}\right), \tag{177}$$

and $\alpha_m < \theta < \beta_m \ (m = 1, 2, \ldots, n)$, where

$$\alpha_m = \frac{2\pi}{l} a_m, \quad \beta_m = \frac{2\pi}{l} b_m \quad (m = 1, 2, \ldots, n). \tag{178}$$

In Chapter I we showed that equation (268) has the solution (318), where constants are determined by equations (319), (322) and (327) and

$$P = \sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m} p(\varphi)\,d\varphi.$$

Thus, equation (177) will have the solution

$$p\left(\frac{l\vartheta}{2\pi}\right)=-\frac{1}{2\pi^2}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\vartheta-\alpha_m}{2}}\right|}\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}\times$$

$$\times f'\left(\frac{l\varphi}{2\pi}\right)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi+(-1)^{n-m+1}\frac{\sum_{m=0}^{n}\gamma_m\sin^{n-m}\frac{\vartheta}{2}\cos^{m}\frac{\vartheta}{2}}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\vartheta-\alpha_m}{2}\sin\frac{\vartheta-\beta_m}{2}\right|}} \qquad (179)$$

$$\alpha_m<\vartheta<\beta_m\ (m=1,2,\ldots,n),$$

where constants $\gamma_0, \gamma_1, \ldots, \gamma_n$ are determined by equations

$$\left.\begin{aligned}
&-\gamma_0+\gamma_2-\gamma_4+\gamma_6-\ldots=\frac{P^0}{2\pi}\sin\sum_{m=1}^{n}\frac{\alpha_m+\beta_m}{4}+\\
&+\frac{1}{2\pi^2}\left(\sin\sum_{m=1}^{n}\frac{\beta_m}{2}\right)\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'\left(\frac{l\varphi}{2\pi}\right)d\varphi,\\
&-\gamma_1+\gamma_3-\gamma_5+\gamma_7-\ldots=\frac{P^0}{2\pi}\cos\sum_{m=1}^{n}\frac{\alpha_m+\beta_m}{4}+\\
&+\frac{1}{2\pi^2}\left(\cos\sum_{m=1}^{n}\frac{\beta_m}{2}\right)\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'\left(\frac{l\varphi}{2\pi}\right)d\varphi,
\end{aligned}\right\} \qquad (180)$$

where

$$P^0=\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}p\left(\frac{l\varphi}{2\pi}\right)d\varphi, \qquad (181)$$

and equations

$$\sum_{m=0}^{n}\gamma_m\int_{\beta_k}^{\alpha_{k+1}}\frac{\sin^{n-m}\frac{\vartheta}{2}\cos^{m}\frac{\vartheta}{2}d\vartheta}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\vartheta-\alpha_m}{2}\sin\frac{\vartheta-\beta_m}{2}\right|}}=$$

$$=(-1)^{n-k}\left[2\frac{f\left(\frac{l\alpha_{k+1}}{2\pi}\right)-f\left(\frac{l\beta_{k+1}}{2\pi}\right)}{l}+\frac{1}{2\pi^2}\int_{\beta_k}^{\alpha_{k+1}}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\vartheta-\beta_m}{2}}{\sin\frac{\vartheta-\alpha_m}{2}}\right|}\times\right.$$

$$\left.\times\left(\sum_{m=1}^{n}\int_{\alpha_m}^{\beta_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\varphi-\alpha_m}{2}}{\sin\frac{\varphi-\beta_m}{2}}\right|}f'\left(\frac{l\varphi}{2\pi}\right)\operatorname{ctg}\frac{\varphi-\vartheta}{2}d\varphi\right)d\vartheta\right], \qquad (182)$$

$$k=1,2,\ldots,n-1.$$

Assuming in (181) $\varphi = \frac{2\pi}{l}t$, we will find

$$P^{\circ} = \frac{2\pi}{l}\sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt = \frac{2\pi}{l}P, \tag{183}$$

where

$$P = \sum_{m=1}^{n}\int_{a_m}^{b_m} p(t)\,dt \tag{184}$$

is compressing force taken over interval $0 \leqslant x \leqslant l$, i.e., one period. Substituting (183) in (180) and assuming in (179), (180) and (182) $\varphi = \frac{2\pi}{l}t,\ \theta = \frac{2\pi}{l}x$, we will find

$$\begin{aligned}
p(x) = {} & \\
= -\frac{1}{\pi l} & \sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-b_m)}{\sin\frac{\pi}{l}(x-a_m)}\right|}\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(t-a_m)}{\sin\frac{\pi}{l}(t-b_m)}\right|}\times \\
& \times f'(t)\operatorname{ctg}\frac{\pi}{l}(t-x)\,dt + \\
& + (-1)^{n-m+1}\frac{\sum\limits_{m=0}^{n}\gamma_m \sin^{n-m}\frac{\pi x}{l}\cos^{m}\frac{\pi x}{l}}{\sqrt{\left|\prod\limits_{m=1}^{n}\sin\frac{\pi}{l}(x-a_m)\sin\frac{\pi}{l}(x-b_m)\right|}}, \\
& a_m < x < b_m\ (m = 1, 2, \ldots, n),
\end{aligned} \tag{185}$$

where $\gamma_0, \gamma_1, \ldots, \gamma_n$ are determined by equations

$$\left.\begin{aligned}
-\gamma_0 + \gamma_2 - \gamma_4 + \gamma_6 - \ldots = {} & \frac{P}{l}\sin\sum_{m=1}^{n}\frac{\pi}{2l}(a_m+b_m) + \\
+ \frac{1}{\pi l}\left(\sin\sum_{m=1}^{n}\frac{\pi b_m}{l}\right)\sum_{m=1}^{n}\int_{a_m}^{b_m} & \sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(t-a_m)}{\sin\frac{\pi}{l}(t-b_m)}\right|}f'(t)\,dt, \\
-\gamma_1 + \gamma_3 - \gamma_5 + \gamma_7 - \ldots = {} & \frac{P}{l}\cos\sum_{m=1}^{n}\frac{\pi}{2l}(a_m+b_m) + \\
+ \frac{1}{\pi l}\left(\cos\sum_{m=1}^{n}\frac{\pi b_m}{l}\right)\sum_{m=1}^{n}\int_{a_m}^{b_m} & \sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(t-a_m)}{\sin\frac{\pi}{l}(t-b_m)}\right|}f'(t)\,dt,
\end{aligned}\right\} \tag{186}$$

and

$$\sum_{m=0}^{n} \gamma_m \int_{b_k}^{a_{k+1}} \frac{\sin^{n-m}\frac{\pi x}{l}\cos^{m}\frac{\pi x}{l}\,dx}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\pi}{l}(x-a_m)\sin\frac{\pi}{l}(x-b_m)\right|}} =$$

$$= \frac{(-1)^{n-k}}{\pi}\Bigg[f(a_{k+1})-f(b_k)+$$

$$+\frac{1}{l}\int_{b_k}^{a_{k+1}}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-b_m)}{\sin\frac{\pi}{l}(x-a_m)}\right|}\left(\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(t-a_m)}{\sin\frac{\pi}{l}(t-b_m)}\right|}\times\right.$$

$$\left.\times f'(t)\operatorname{ctg}\frac{\pi}{l}(t-x)\,dt\right)dx\Bigg], \quad k=1, 2, \ldots, n-1. \tag{187}$$

So that pressure $p(x)$, determined by formula (185), is limited everywhere in the region of contact, including points $x = a_m$ and $x=b_m\ (m=1, 2, \ldots, n)$, conditions

$$\sum_{m=0}^{n}\gamma_m\sin^{n-m}\frac{\pi b_k}{l}\cos^{m}\frac{\pi b_k}{l}=0,$$

$$\sum_{m=0}^{n}\gamma_m\sin^{n-m}\frac{\pi a_k}{l}\cos^{m}\frac{\pi a_k}{l}=$$

$$=(-1)^{n-k+1}\frac{1}{\pi l}\left|\prod_{m=1}^{n}\sin\frac{\pi}{l}(a_k-b_m)\right|\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(t-a_m)}{\sin\frac{\pi}{l}(t-b_m)}\right|}\times$$

$$\times f'(t)\operatorname{ctg}\frac{\pi}{l}(t-a_k)\,dt, \qquad k=1, 2, \ldots, n. \tag{188}$$

must be fulfilled. Formulas (185), (186), (187) and (188) completely solve the contact problem examined in this chapter by determining the region of contact and pressure $p(x)$. Equations (186), (187) and (188) determine the abscissas of beginning and ends of sections of contact $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ and constants $\gamma_0, \gamma_1, \ldots, \gamma_n$, after which formula (185) determines pressure $p(x)$ in the region of contact.

If pressing on the lower elastic half-plane in interval $0<x<l$, i.e., within one period, are n separate bodies, and compressing forces $P_1, P_2, \ldots, P_n$, which act on each of these bodies are assigned, then instead of equations (187) we will have equations

$$\sum_{m=0}^{n}\gamma_m\int_{a_k}^{b_k}\frac{\sin^{n-m}\frac{\pi x}{l}\cos^{m}\frac{\pi x}{l}\,dx}{\sqrt{\left|\prod_{m=1}^{n}\sin\frac{\pi}{l}(x-a_m)\sin\frac{\pi}{l}(x-b_m)\right|}}=$$

$$=(-1)^{n-k+1}\left[P_k+\frac{1}{\pi l}\int_{a_k}^{b_k}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(x-b_m)}{\sin\frac{\pi}{l}(x-a_m)}\right|}\times\right.$$

$$\left.\times\left(\sum_{m=1}^{n}\int_{a_m}^{b_m}\sqrt{\left|\prod_{m=1}^{n}\frac{\sin\frac{\pi}{l}(t-a_m)}{\sin\frac{\pi}{l}(t-b_m)}\right|}f'(t)\operatorname{ctg}\frac{\pi}{l}(t-x)\,dt\right)dx\right], \qquad (189)$$

$$k=1, 2, \ldots, n-1,$$

which express, according to (185), conditions

$$\int_{a_k}^{b_k}p(x)\,dx=P_k \quad (k=1, 2, \ldots, n-1),$$

and in equations (186) P must be in this case understood as the sum

$$P=P_1+P_2+\ldots+P_n. \qquad (190)$$

Let us also note that if some of the boundaries of sections of contact are predetermined beforehand by the configuration of the compressible bodies, as in the case of the pressure of the stamp with right angles on the elastic half-plane, then the appropriate equations of (188) must be dropped.

In conclusion of this section let us examine the simplest examples.

1) Let us examine the pressure on the lower elastic half-plane of the stamp having outlines shown in Fig. 22. In this case $n=1$, $a_1=\frac{l}{2}-a$, $a_2=\frac{l}{2}+a$, $f(x)=\text{const.}$ when $\frac{l}{2}-a<x<\frac{l}{2}+a$, and equation (177) takes the form

$$\int_{\pi-\alpha}^{\pi+\alpha}p\left(\frac{l\varphi}{2\pi}\right)\ln\frac{1}{2\left|\sin\frac{\varphi-\theta}{2}\right|}\,d\varphi=\text{const.} \qquad (191)$$

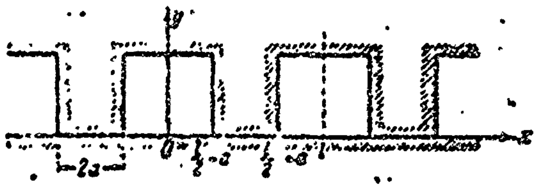

Fig. 22.

when $\pi-\alpha<\theta<\pi+\alpha$,

$$\alpha=\frac{2\pi a}{l}.$$

In Chapter I we showed that equation

$$\int_{\pi-\alpha}^{\pi+\alpha} p(\varphi)\ln\frac{1}{2\left|\sin\frac{\varphi-\theta}{2}\right|}d\varphi=\mathrm{const.},\quad \pi-\alpha<\theta<\pi+\alpha,$$

has the solution

$$p(\pi+\theta)=\frac{P^{*}\sqrt{2}\cos\frac{\theta}{2}}{2\pi\sqrt{\cos\theta-\cos\alpha}},\quad -\alpha<\theta<\alpha$$

(see formula (338) of Chapter I). Thus, equation (191) has the solution

$$p\left(\frac{l}{2}+\frac{l\theta}{2\pi}\right)=\frac{P^{*}\sqrt{2}\cos\frac{\theta}{2}}{2\pi\sqrt{\cos\theta-\cos\alpha}},\quad -\alpha<\theta<\alpha. \tag{192}$$

Substituting (183) into (192) and assuming according to (176) and (178) $\theta=\frac{2\pi}{l}x$, $\alpha=\frac{2\pi}{l}a$, we will find

$$p\left(\frac{l}{2}+x\right)=\frac{P\sqrt{2}\cos\frac{\pi x}{l}}{l\sqrt{\cos\frac{2\pi x}{l}-\cos\frac{2\pi a}{l}}},\quad -a<x<a. \tag{193}$$

Figure 23 shows graphs of pressure p under the stamp, which are plotted in accordance with formula (193) for different values of ratio $\frac{a}{l}$. When $l=2a\left(\frac{a}{l}=0{,}5\right)$, as one can see from Fig. 22, the bases of the stamps merge, and we arrive at a pressure evenly distributed under the stamp $p=\frac{P}{2a}$. When $l\to\infty\left(\frac{a}{l}\to 0\right)$, we arrive at the case of one section of the contact examined in § 3, and we will obtain the same graph of pressure $p(x)$ which is in Fig. 13.

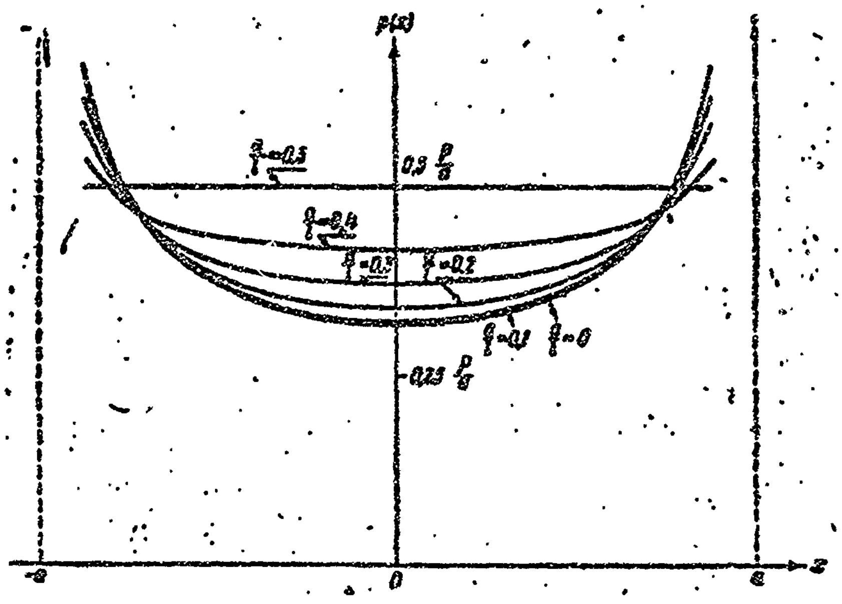

Fig. 23.

2) Let us assume that now (Fig. 24)

$$f_1(x)=\frac{Al^2}{2\pi^2}\cos^2\frac{\pi x}{l},\quad f_2(x)=0. \tag{194}$$

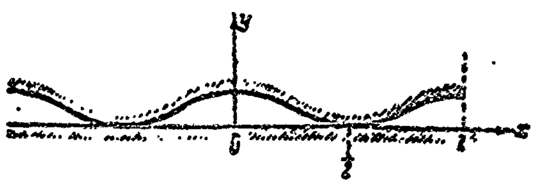

Fig. 24.

Let us note that in this case $f_1''\left(\frac{l}{2}\right)=A$, i.e., constant A in formula (194) determines the curvature at the point of contact. Substituting (194) into (175), we find

$$f(x)=-\frac{Al^2}{2\pi^2(\vartheta_1+\vartheta_2)}\cos^2\frac{\pi x}{l}+\text{const.},$$

and equation (177) will for this case have the form

$$\int_{\pi-\alpha}^{\pi+\alpha} p\left(\frac{l\varphi}{2\pi}\right)\ln\frac{1}{2\left|\sin\frac{\varphi-\vartheta}{2}\right|}\,d\varphi=-\frac{Al}{\pi(\vartheta_1+\vartheta_2)}\cos^2\frac{\vartheta}{2}+\text{const.}, \tag{195}$$

$$\pi-\alpha<\vartheta<\pi+\alpha,\quad \alpha=\frac{2\pi a}{l},$$

if by a we designate the half-width of the section of contact.

In Chapter I we showed that equation

$$\int_{\pi-\alpha}^{\pi+\alpha} p(\varphi) \ln \frac{1}{2\left|\sin\frac{\varphi-\vartheta}{2}\right|} d\varphi = -A\cos^2\frac{\vartheta}{2} + \text{const.}$$

has the solution

$$p(\pi+\vartheta) = \frac{\sqrt{2}A}{2\pi}\cos\frac{\vartheta}{2}\sqrt{\cos\vartheta-\cos\alpha}, \quad \sin\frac{\alpha}{2} = \sqrt{\frac{P}{A}}$$

(see formulas (354) and (355) of Chapter I).

Thus, equation (195) will have the solution

$$p\left(\frac{l}{2}+\frac{l\vartheta}{2\pi}\right) = \frac{Al\sqrt{2}}{2\pi^2(\vartheta_1+\vartheta_2)}\cos\frac{\vartheta}{2}\sqrt{\cos\vartheta-\cos\alpha}, \quad -\alpha<\vartheta<\alpha, \tag{196}$$

$$\sin\frac{\alpha}{2} = \sqrt{\frac{\pi P^*(\vartheta_1+\vartheta_2)}{Al}}. \tag{197}$$

Substituting (183) into (197) and assuming in (196) and (197) $\vartheta = \frac{2\pi}{l}x$, $\alpha = \frac{2\pi}{l}a$ according to (176) and (178), we will find

$$p\left(\frac{l}{2}+x\right) = \frac{Al\sqrt{2}}{2\pi^2(\vartheta_1+\vartheta_2)}\cos\frac{\pi x}{l}\sqrt{\cos\frac{2\pi x}{l}-\cos\frac{2\pi a}{l}}, \tag{198}$$

$$-a<x<a,$$

$$a = \frac{l}{\pi}\arcsin\frac{\pi}{l}\sqrt{\frac{2P(\vartheta_1+\vartheta_2)}{A}}. \tag{199}$$

Formula (199) determines the half-width of the section of contact a, and formula (198) – the pressure p in the region of contact.

As can be seen from formula (199), when

$$l = \pi\sqrt{\frac{2P(\vartheta_1+\vartheta_2)}{A}} = l_0, \tag{200}$$

$a=\frac{l_0}{2}$, i.e., complete contact of compressible bodies occurs along the entire Ox axis. Using designation (200)

$$\pi\sqrt{\frac{2P(\theta_1+\theta_2)}{A}}=l_0,$$

it is possible to give formulas (198) and (199) the form

$$p\left(\frac{l}{2}+x\right)=\sqrt{2}\,\frac{l}{l_0}\cos\frac{\pi x}{l}\sqrt{\cos\frac{2\pi x}{l}-\cos\frac{2\pi a}{l}}\,\frac{P}{l_0},\tag{201}$$
$$-a<x<a,\quad \frac{a}{l}=\frac{1}{\pi}\arcsin\frac{l_0}{l}.$$

Setting different values of the ratio a/l and calculating further by formulas (201) at first l_0/l, and then p, we obtain graphs of pressure p in the region of contact depicted in Fig. 25. These graphs show the dependence of pressure p in the region of contact on the period l at fixed l_0, i.e., according to (200) at fixed elastic constants, compressing force P and at fixed constant A, i.e., at fixed curvature at the point of initial contact. In the maximum case when $l = l_0$, $a/l = 0.5$ (complete contact of elastic bodies along the entire Ox axis), the pressure p changes according to sinusoid. In the other maximum case when $l\to\infty, a/l\to 0$, we arrive at the initial contact of the compressible bodies at one point and obtain the distribution of pressure in the region of contact along the ellipse, already obtained earlier by us in § 2. Curves in Fig. 25 also give a clear representation about the mutual effect of pressure on the neighboring sections.

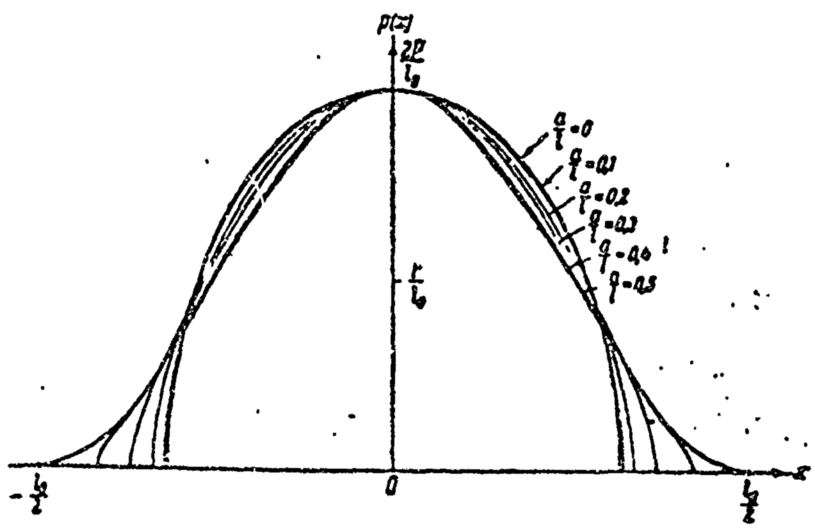

Fig. 25.

§ 6. Contact Problem in the Presence of Frictional Forces

Thus far we assumed that friction between compressible bodies is absent, and that in the region of contact only normal pressure $p(x)$ acts on the compressed bodies. Let us consider now the case when between the compressible bodies friction[1] takes place. Let us assume that compressible bodies are found on the threshold of equilibrium and in the region of contact between normal and tangential stresses there is the relation

$$\tau_{xy} = k\sigma_{yy}, \tag{202}$$

where k – coefficient of friction. Thus, at the point of contact with the abscissa x in the presence of normal pressure $p(x)$ the tangential stress $t(x) = kp(x)$ will also act. Due to the presence of these tangential stresses, end points of compressible bodies with the abscissa x will accomplish additional elastic displacements in the direction of the Oy axis, which we will designate by v_1^* and v_2^*. As is known from the theory of elasticity, these displacements will be connected with the tangential stress $t(x) = kp(x)$ by relation

$$\frac{dv_1^*}{dx} = \frac{t(x)}{G_1}, \qquad \frac{dv_2^*}{dx} = \frac{t(x)}{G_2}, \tag{203}$$

or

$$\frac{dv_1^*}{dx} = \frac{k}{G_1} p(x), \quad \frac{dv_2^*}{dx} = \frac{k}{G_2} p(x), \tag{204}$$

where G_1 and G_2 are shear moduli of compressible materials. Integrating (204) with respect to x, we will find

[1]This problem was solved by N. I. Muskhelishvili, L. A. Galin and in part by N. I. Glagolev. See Academician N. I. Muskhelishvili, Singular integral equations; L. A. Galin, Mixed problems of the theory of elasticity with frictional forces for a half-plane, Reports of the Academy of Sciences of the USSR, Vol. XXIX, No. 3, 1943; N. I. Glagolev, Elastic stresses along bases of a dam, Reports of the Academy of Sciences of the USSR, Vol. XXXIV, No. 7, 1942.

$$v_1^* = \frac{k}{G_1}\int_0^x p(t)\,dt + \text{const.}, \quad v_2^* = \frac{k}{G_2}\int_0^x p(t)\,dt + \text{const.} \tag{205}$$

In order to obtain the integral equation of the problem, into condition (3) instead of displacements v_1 and v_2, determined by formulas (9) and (10), it is necessary to substitute displacements $v_1 + v_1^*$ and $v_2 + v_2^*$. Instead of the integral equation (12) let us obtain equation

$$k\frac{G_1+G_2}{G_1G_2}\int_0^x p(t)\,dt + (\theta_1+\theta_2)\int_S p(t)\ln\frac{1}{|t-x|}\,dt = c - f_1(x) - f_2(x) \text{ on } S, \tag{206}$$

or

$$\int_0^x p(t)\,dt + \nu\int_S p(t)\ln\frac{1}{|t-x|}\,dt = \nu f(x) \text{ on } S, \tag{207}$$

where

$$\nu = \frac{(\theta_1+\theta_2)\,G_1G_2}{k(G_1+G_2)}, \tag{208}$$

$$f(x) = \frac{c - f_1(x) - f_2(x)}{\theta_1+\theta_2}. \tag{209}$$

For simplicity we will subsequently assume that the region of contact S is composed of one section. If by $2a$ we designate the length of this section and dispose the origin of the coordinates in the center of the section of contact, then equation (206) will have the form

$$\int_0^x p(t)\,dt + \nu\int_{-a}^{a} p(t)\ln\frac{1}{|t-x|}\,dt = \nu f(x), \quad -a < x < a. \tag{210}$$

Using formulas (391) and (392) of Chapter I for the solution of the integral equation (356) of Chapter I, we will obtain the following solution to equation (210):

$$p(x)=\frac{\sin\pi\gamma\cos\pi\gamma}{\pi}f'(x)-\frac{\frac{\cos^2\pi\gamma}{\pi}\int\limits_{-a}^{a}\frac{(a+t)^{\frac{1}{2}+\gamma}(a-t)^{\frac{1}{2}-\gamma}f'(t)\,dt}{t-x}-P\cos\pi\gamma}{\pi(a+x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma}},\quad -a<x<a, \tag{211}$$

where

$$P=\int\limits_{-a}^{a}p(t)\,dt, \tag{212}$$

i.e., constitutes the compressing force, and

$$\gamma=\frac{1}{\pi}\operatorname{arc\,tg}\frac{1}{\pi\nu}. \tag{213}$$

Substituting (208) into (213), we will find

$$\gamma=\frac{1}{\pi}\operatorname{arc\,tg}\frac{k(\theta_1+\theta_2)}{\pi(\vartheta_1+\vartheta_2)\theta_1\theta_2}. \tag{214}$$

So that the pressure $p(x)$, determined by formula (211), was limited everywhere in the region of contact, including points $x = -a$ and $x = a$, there must be fulfilled these conditions

$$\cos\pi\gamma\int\limits_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\frac{1}{2}+\gamma}f'(t)\,dt=-\pi P \tag{215}$$

and

$$\int\limits_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\frac{1}{2}+\gamma}f'(t)\,dt=-\int\limits_{-a}^{a}\left(\frac{a-t}{a+t}\right)^{\frac{1}{2}-\gamma}f'(t)\,dt. \tag{216}$$

Condition (216) can be given the form

$$\int\limits_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\left(\frac{\sqrt{a+t}}{\sqrt{a-t}}+\frac{\sqrt{a-t}}{\sqrt{a+t}}\right)f'(t)\,dt=2a\int\limits_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}}=0,$$

or

$$\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}}=0. \qquad (217)$$

Using the relation (216), condition (215) can be given the form

$$\cos\pi\gamma\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\left(\frac{\sqrt{a+t}}{\sqrt{a-t}}-\frac{\sqrt{a-t}}{\sqrt{a+t}}\right)f'(t)\,dt=-2\pi P,$$

or

$$\cos\pi\gamma\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{t f'(t)\,dt}{\sqrt{a^2-t^2}}=-\pi P. \qquad (218)$$

Using relations (217) and (218), we will find

$$\int_{-a}^{a}\frac{(a+t)^{\frac{1}{2}+\gamma}(a-t)^{\frac{1}{2}-\gamma}f'(t)\,dt}{t-x}=\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{(a^2-t^2)f'(t)\,dt}{\sqrt{a^2-t^2}(t-x)}=$$

$$=(a^2-x^2)\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}(t-x)}-x\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}}-$$

$$-\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{t f'(t)\,dt}{\sqrt{a^2-t^2}}=(a^2-x^2)\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}(t-x)}+\frac{\pi P}{\cos\pi\gamma}. \qquad (219)$$

Substituting (219) into (211), we will find

$$p(x)=\frac{\sin\pi\gamma\cos\pi\gamma}{\pi}f'(x)-\frac{\cos^2\pi\gamma}{\pi^2}(a+x)^{\frac{1}{2}-\gamma}(a-x)^{\frac{1}{2}+\gamma}\times$$

$$\times\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{\gamma}\frac{f'(t)\,dt}{\sqrt{a^2-t^2}(t-x)}, \qquad -a<x<a. \qquad (220)$$

Condition (217) predetermines the selection of the origin of the coordinates, i.e., determines the position of the section of contact, equation (218) determines the half-width of the section of contact a, and formula (220) determines pressure $p(x)$ in the region of contact. Thus, formulas (214), (217), (218) and (220) completely solve the problem by determining the region of contact and pressure $p(x)$. When $k = 0$, i.e., with the absence of friction $\gamma = 0$ according to

(214). Assuming $\gamma = 0$ in (217), (218) and (220), we will obtain formulas (32), (33) and (34), derived by us earlier on the assumption of the absence of friction.

If the region of contact is predetermined beforehand by the configuration of the compressible bodies, conditions (215) and (216) are dropped, and pressure $p(x)$ is determined directly by formula (211). Thus, for example, if a stamp with right angles, which is under the action of the normal force P and tangent force T, presses on the elastic half-plane (Fig. 26a), then $f(x)$ = const when $-a < x < a$, and formula (211) gives

$$p(x) = \frac{P\cos\pi\gamma}{\pi(a+x)^{\frac{1}{2}+\gamma}(a-x)^{\frac{1}{2}-\gamma}}, \quad -a < x < a. \tag{221}$$

Figure 26b shows the distribution of pressure $p(x)$ under the stamp for $\gamma = 0.05$. In the absence of friction, i.e., when $\gamma = 0$, formula (221) passes into formula (116) derived by us earlier for this case.

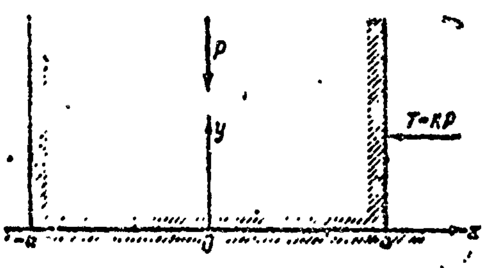

Fig. 26a.

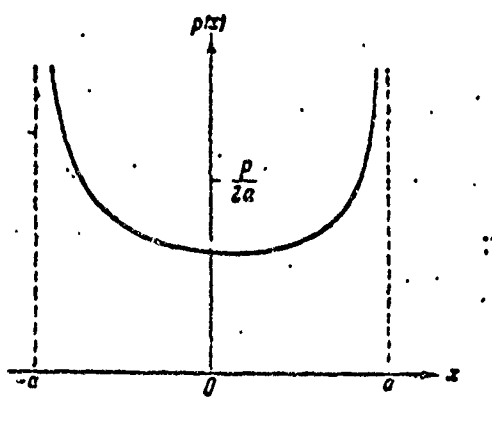

Fig. 26b.

§ 7. Compression of Elastic Bodies Limited by Cylindrical Surfaces the Radii of Which Are Almost Equal

Let us now examine the problem about the compression of two elastic bodies, one of which has the form of a circular cylinder, and the other has a circular cylindrical cut (Fig. 27). Let us designate by r_1 and r_2 radii of cylindrical surfaces limiting the compressible bodies. If these radii are minutely different from one another, then with the compression of elastic bodies the contact between their surfaces can spread over a considerable part of these surfaces, and thus, the theory of compression, discussed by us in § 1 of this chapter, will already be inapplicable. Below we derive the equation determining in this case the distribution of pressure over the surface of compression.

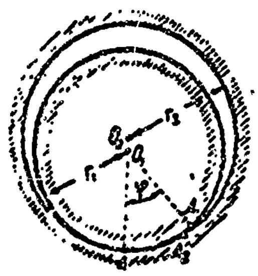

Fig. 27.

Let us assume that A_1 and A_2 are two points of elastic bodies, touching with compression (Fig. 27 and Fig. 28), A is the point of initial contact of the compressible bodies, and ϕ is the angle of AO_1A_1. Let us assume that further $\overline{A_1A_1'}$ and $\overline{A_2A_2'}$ are elastic displacements of points A_1 and A_2. Then the segment $A_1'A_2'$ will constitute the approach of the elastic bodies with compression, due to which contact between points A_1 and A_2 is carried out. We assume here that resultants of compressing forces pass through the point of initial contact of compressible bodies A and centers O_1 and O_2 so that a relative turn of the compressible bodies does not occur, and only relative forward displacement of the bodies with compression appears. Let us designate further by u_{1r} and u_{2r} the normal elastic displacements of points A_1 and A_2 (segments $\overline{A_1A_1''}$ and $\overline{A_2A_2''}$ in Fig. 28)

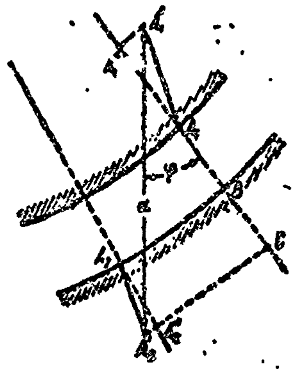

Fig. 28.

and by α – the approach of the bodies with compression In view of the smallness of the elastic displacements, it is possible to take segment $\overline{BC}$ in Fig. 28 equal to segment $\overline{A_2A_2''}$. Then from Fig. 28 we will find

$$u_{1\varphi}+u_{2\varphi}=\alpha\cos\varphi-\overline{A_1B}. \tag{222}$$

Let us calculate now segment $\overline{A_1B}$. Since point A_1 lies on the internal and point B on the external cylindrical surface, $\overline{O_1A_1} = r_1$, $\overline{O_2B} = r_2$ (Fig. 29). As can be seen from Fig. 29, it is possible to assume approximately

$$\overline{O_2B}-\overline{O_1B}=\overline{O_1O_2}\cos\varphi, \tag{223}$$

since the difference in radii $r_2 - r_1 = \overline{O_1C}$ on assumption is minute. The relation (223) can be given the form

$$r_2-(r_1+\overline{A_1B})=(r_2-r_1)\cos\varphi,$$

whence

$$\overline{A_1B}=(r_2-r_1)(1-\cos\varphi). \tag{224}$$

Substituting (224) into (222), we will obtain the relation

$$u_{1\varphi}+u_{2\varphi}=\alpha\cos\varphi-(r_2-r_1)(1-\cos\varphi), \tag{225}$$

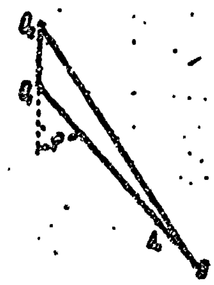

Fig. 29.

which should take place in the region of contact. We will subsequently assume that surfaces of compressible bodies are perfectly smooth. Then acting on the surface of contact will be only normal pressure, which we designate by $p(\phi)$. Let us express now displacements u_{1r} and u_{2r}, which enter into the relation (225), in terms of the unknown pressure $p(\phi)$. Let us consider for this purpose the action of the concentrated pressure on the elastic body limited by circular cylindrical surface (Fig. 30). As is known, in this case of forces P, shown in Fig. 30, cause in the elastic body simple radial distributions of stresses and uniform expansion[1] $\frac{P}{\pi r}$. Under the impact of these forces, point A accomplishes radial elastic displacement in the direction of center O:

$$u_r = P\left[-2\vartheta\left(1+\cos\varphi\ln\operatorname{tg}\frac{|\varphi|}{2}\right)+\varkappa\sin|\varphi|\right], \tag{226}$$

where

$$\vartheta = \frac{\lambda+2\mu}{4\pi\mu(\lambda+\mu)},$$
$$\varkappa = \frac{1}{4(\lambda+\mu)},$$

λ and μ — elastic constants of the compressible body. Analogously, point A of the infinite elastic body with a circular cylindrical cut under the impact of forces P shown in Fig. 31, accomplishes a radial elastic displacement in a direction from the center O:

$$u_r = P\left(-2\vartheta\cos\varphi\ln\operatorname{tg}\frac{|\varphi|}{2}+\varkappa\sin|\varphi|\right). \tag{227}$$

[1]See Timoshenko S. P., Theory of elasticity, 1937, p. 118.

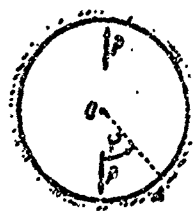

Fig. 30.   Fig. 31.

Let us now calculate the displacement u_{1r}, which enters into the relation (225). If $p(\phi)$ is the normal pressure on the surface of contact, then on the element KL of the internal cylinder (Fig. 32) force $p(\varphi')r_1 d\varphi'$ will act. Jointly with the same force applied to the diametrically opposite point of the cylinder, this force will cause at point A' a radial displacement du_{1r}, which according to (226) will equal

$$du_{1r} = p(\varphi')r_1\left\{-2\vartheta_1\left[1+\cos(\varphi-\varphi')\ln\operatorname{tg}\frac{|\varphi-\varphi'|}{2}\right]+\varkappa_1\sin|\varphi-\varphi'|\right\}d\varphi'. \qquad (228)$$

Let us assume that further, as is shown in Fig. 32, that region of contact corresponds to a change in angle ϕ within limits of $-\phi_0$ to ϕ_0. Then we will find the complete radial displacement of point A' by integrating the expression for the elementary displacement du_{1r} with respect to ϕ' within limits of $-\phi_0$ to ϕ_0:

$$u_{1r} = \int_{-\varphi_0}^{\varphi_0} p(\varphi')r_1\left\{-2\vartheta_1\left[1+\cos(\varphi-\varphi')\ln\operatorname{tg}\frac{|\varphi-\varphi'|}{2}\right]+\varkappa_1\sin|\varphi-\varphi'|\right\}d\varphi'. \qquad (229)$$

With derivation of formula (229) we assumed that the external compressing forces, which act on the internal cylinder, are distributed along its surface symmetrical to the pressure which they cause in the region of contact. This assumption is permissible due to the fact that the pressure in the region of contact depends little on how external compressing forces are applied, and with solution of the

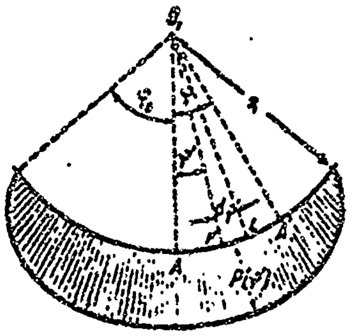

Fig. 32.

contact problem it is possible to be distracted from the distribution of external compressing forces similar to the way we did this in § 1 of this chapter with the derivation of the fundamental equation of the contact problem.

Literally repeating the reasonings conducted by us in the derivation of formula (229), we will find in conformity with (227) the following expression for the displacement u_{2r}:

$$u_{2r} = \int_{-\varphi_0}^{\varphi_0} p(\varphi') r_2 \Big[-2\vartheta_2 \cos(\varphi - \varphi') \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} + \\ + \varkappa_2 \sin|\varphi - \varphi'| \Big] d\varphi'. \qquad (230)$$

Substituting (229) and (230) into (225), we will obtain the equation[1]

$$-2(\vartheta_1 r_1 + \vartheta_2 r_2) \int_{-\varphi_0}^{\varphi_0} p(\varphi') \cos(\varphi - \varphi') \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} d\varphi' + \\ + (\varkappa_1 r_1 + \varkappa_2 r_2) \int_{-\varphi_0}^{\varphi_0} p(\varphi') \sin|\varphi - \varphi'| d\varphi' - 2\vartheta_1 r_1 \int_{-\varphi_0}^{\varphi_0} p(\varphi') d\varphi' = \\ = a \cos\varphi - (r_2 - r_1)(1 - \cos\varphi), \\ -\varphi_0 < \varphi < \varphi_0, \qquad (231)$$

where

[1]For the numerical solution of equation (231) the method of finite differences is very convenient (see Appendix 2).

$$\vartheta_1 = \frac{\lambda_1 + 2\mu_1}{4\pi\mu_1(\lambda_1 + \mu_1)}, \quad \vartheta_2 = \frac{\lambda_2 + 2\mu_2}{4\pi\mu_2(\lambda_2 + \mu_2)},$$

$$\varkappa_1 = \frac{1}{4(\lambda_1 + \mu_1)}, \quad \varkappa_2 = \frac{1}{4(\lambda_2 + \mu_2)},$$

λ_1, μ_1 and λ_2, μ_2 – elastic constants of the compressible bodies. Let us now consider the function of angle ϕ:

$$J(\varphi) = 2(\vartheta_1 r_1 + \vartheta_2 r_2) \int_{-\varphi_0}^{\varphi_0} p(\varphi') \sin(\varphi - \varphi') \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} \, d\varphi' +$$
$$+ (\varkappa_1 r_1 + \varkappa_2 r_2) \left[\int_{-\varphi_0}^{\varphi} p(\varphi') \cos(\varphi - \varphi') \, d\varphi' - \right.$$
$$\left. - \int_{\varphi}^{\varphi_0} p(\varphi') \cos(\varphi - \varphi') \, d\varphi' \right]. \qquad (232)$$

Let us calculate the derivative $J'(\phi)$. Since when $\phi > \varphi'$

$$\frac{d}{d\varphi} \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} = \frac{d}{d\varphi} \ln \operatorname{tg} \frac{\varphi - \varphi'}{2} = \frac{1}{\sin(\varphi - \varphi')}$$

and when $\phi < \phi'$

$$\frac{d}{d\varphi} \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} = \frac{d}{d\varphi} \ln \operatorname{tg} \frac{\varphi' - \varphi}{2} = \frac{1}{\sin(\varphi - \varphi')},$$

we will have

$$\frac{d}{d\varphi} \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} = \frac{1}{\sin |\varphi - \varphi'|}. \qquad (233)$$

Taking into account formula (230), from (232) we will find

$$J'(\varphi) = 2(\vartheta_1 r_1 + \vartheta_2 r_2) \left[\int_{-\varphi_0}^{\varphi_0} p(\varphi') \cos(\varphi - \varphi') \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} \, d\varphi' + \right.$$
$$\left. + \int_{-\varphi_0}^{\varphi_0} p(\varphi') \, d\varphi' \right] - (\varkappa_1 r_1 + \varkappa_2 r_2) \left[\int_{-\varphi_0}^{\varphi} p(\varphi') \sin(\varphi - \varphi') \, d\varphi' + \right.$$
$$\left. + \int_{\varphi}^{\varphi_0} p(\varphi') \sin(\varphi' - \varphi) \, d\varphi' - 2p(\varphi) \right],$$

whence

$$-2(\vartheta_1 r_2+\vartheta_2 r_1)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\cos(\varphi-\varphi')\ln\operatorname{tg}\frac{|\varphi-\varphi'|}{2}\,d\varphi'+$$
$$+(\varkappa_1 r_2+\varkappa_2 r_1)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\sin|\varphi-\varphi'|\,d\varphi'=$$
$$=2(\vartheta_1 r_2+\vartheta_2 r_1)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'+2(\varkappa_1 r_2+\varkappa_2 r_1)\,p(\varphi)-J'(\varphi). \tag{234}$$

Substituting (234) into (231), we will find

$$2\vartheta_2 r_2\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'+2(\varkappa_1 r_2+\varkappa_2 r_1)\,p(\varphi)-J'(\varphi)=$$
$$=\varepsilon\cos\varphi-(r_2-r_1)(1-\cos\varphi),$$

whence

$$J'(\varphi)=2(\varkappa_1 r_2+\varkappa_2 r_1)\,p(\varphi)+r_2-r_1+$$
$$+2\vartheta_2 r_2\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'-(r_2-r_1+\varepsilon)\cos\varphi. \tag{235}$$

Integrating both sides of equality (235) with respect to ϕ, we will find

$$J(\varphi)=2(\varkappa_1 r_2+\varkappa_2 r_1)\int_{0}^{\varphi} p(\varphi')\,d\varphi'+$$
$$+\left[r_2-r_1+2\vartheta_2 r_2\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi\right]\varphi-(r_2-r_1+\varepsilon)\sin\varphi+2\beta, \tag{236}$$

where β – arbitrary constant.

Differentiating both sides of relation (231) with respect to ϕ and taking into account (233), we will obtain

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\left[\int_{-\varphi_0}^{\varphi_0} p(\varphi')\sin(\varphi-\varphi')\ln\operatorname{tg}\frac{|\varphi-\varphi'|}{2}\,d\varphi'-\right.$$

$$\left.-\int_{-\varphi_0}^{\varphi_0} p(\varphi')\operatorname{tg}(\varphi-\varphi'\right]d\varphi+(\varkappa_2 r_1+\varkappa_1 r_2)\frac{d}{d\varphi}\left[\int_{-\varphi_0}^{\varphi} p(\varphi')\sin(\varphi-\varphi')\,d\varphi'+\right.$$

$$\left.+\int_{\varphi}^{\varphi_0} p(\varphi')\sin(\varphi'-\varphi)\,d\varphi\right]=-(a+r_2-r_1)\sin\varphi,$$

whence

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\sin(\varphi-\varphi')\ln\operatorname{tg}\frac{|\varphi-\varphi'|}{2}\,d\varphi'+$$

$$+(\varkappa_2 r_1+\varkappa_1 r_2)\left[\int_{-\varphi_0}^{\varphi} p(\varphi')\cos(\varphi-\varphi')\,d\varphi'-\int_{\varphi}^{\varphi_0} p(\varphi')\cos(\varphi-\varphi')\,d\varphi'\right]=$$

$$=2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\operatorname{ctg}(\varphi-\varphi')\,d\varphi'-(a+r_2-r_1)\sin\varphi. \qquad (237)$$

Substituting (237) into (232), we will find

$$J(\varphi)=2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\operatorname{ctg}(\varphi-\varphi')\,d\varphi'-(a+r_2-r_1)\sin\varphi. \qquad (238)$$

Substituting (238) into (236), we will obtain the equation

$$(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\operatorname{ctg}(\varphi-\varphi')\,d\varphi-(\varkappa_2 r_1+\varkappa_1 r_2)\times$$

$$\times\int_{0}^{\varphi} p(\varphi')\,d\varphi'=\beta+\gamma_0\varphi,\quad -\varphi_0<\varphi<\varphi_0, \qquad (239)$$

where

$$\gamma_0=\vartheta_2 r_2\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'+\frac{1}{2}(r_2-r_1). \qquad (240)$$

Let us produce in equation (239) the change in variables, having set

$$\operatorname{tg}\varphi=x,\quad \operatorname{tg}\varphi'=t. \qquad (241)$$

Let us find

$$\operatorname{ctg}(\varphi-\varphi')=\frac{1+\operatorname{tg}\varphi\operatorname{tg}\varphi'}{\operatorname{tg}\varphi-\operatorname{tg}\varphi'}=\frac{1+tx}{x-t},\quad d\varphi'=\frac{dt}{1+t^2},$$

and, thus, equation (239) will take the form

$$\begin{aligned}(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-a}^{a} p(\operatorname{arctg} t)\,\frac{1+tx}{(1+t^2)(x-t)}\,dt-\\-(\varkappa_1 r_1+\varkappa_2 r_2)\int_{0}^{x}\frac{p(\operatorname{arctg} t)}{1+t^2}\,dt=\beta+\gamma_0\operatorname{arctg} x,\qquad -a<x<a,\end{aligned}\tag{242}$$

where

$$a=\operatorname{tg}\varphi_0.\tag{243}$$

Using the identity

$$\frac{1+tx}{(x-t)(1+t^2)}=\frac{1}{x-t}+\frac{t}{1+t^2},$$

it is possible to give to equation 242 (form)

$$\begin{aligned}(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-a}^{a}\frac{p(\operatorname{arctg} t)}{x-t}\,dt-(\varkappa_1 r_1+\varkappa_2 r_2)\int_{0}^{x}\frac{p(\operatorname{arctg} t)}{1+t^2}\,dt=\\=\beta-(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-a}^{a}\frac{p(\operatorname{arctg} t)\,t}{1+t^2}\,dt+\gamma_0\operatorname{arctg} x,\qquad -a<x<a.\end{aligned}\tag{244}$$

Assuming further

$$\int_{0}^{x}\frac{p(\operatorname{arctg} t)}{1+t^2}\,dt=g_0(x),\tag{245}$$

we will find

$$p(\operatorname{arctg} x)=(1+x^2)\,g_0'(x).\tag{246}$$

Substituting (246) into (244), we will have

$$(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-a}^{a}\frac{1+t^2}{x-t}\,g_0'(t)\,dt-(\varkappa_1 r_1+\varkappa_2 r_2)\,g_0(x)=$$

$$=\beta-(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-a}^{a} g_0'(t)\,t\,dt+\gamma_0\operatorname{arctg} x,\qquad -a<x<a. \tag{247}$$

Using identity

$$\frac{1+t^2}{x-t}=\frac{1+x^2}{x-t}-t-x,$$

let us give to equation (247) the form

$$(\vartheta_1 r_1+\vartheta_2 r_2)(1+x^2)\int_{-a}^{a}\frac{g_0'(t)\,dt}{x-t}-(\varkappa_1 r_1+\varkappa_2 r_2)\,g_0(x)=$$

$$=\beta+(\vartheta_1 r_1+\vartheta_2 r_2)\,x\int_{-a}^{a} g_0'(t)\,dt+\gamma_0\operatorname{arctg} x,\qquad -a<x<a. \tag{248}$$

Assuming further in equation (248)

$$g_0(x)=g(x)-\frac{\beta}{\varkappa_1 r_1+\varkappa_2 r_2}, \tag{249}$$

we will have

$$(\vartheta_1 r_1+\vartheta_2 r_2)(1+x^2)\int_{-a}^{a}\frac{g'(t)\,dt}{x-t}-(\varkappa_1 r_1+\varkappa_2 r_2)\,g(x)=$$

$$=(\vartheta_1 r_1+\vartheta_2 r_2)\,x\,[g(a)-g(-a)]+\gamma_0\operatorname{arctg} x,$$

or

$$\int_{-a}^{a}\frac{g'(t)\,dt}{t-x}+\frac{\varkappa_1 r_1+\varkappa_2 r_2}{(\vartheta_1 r_1+\vartheta_2 r_2)(1+x^2)}\,g(x)=$$

$$=\frac{[g(-a)-g(a)]\,x}{1+x^2}-\gamma\,\frac{\operatorname{arctg} x}{1+x^2},\qquad -a<x<a, \tag{250}$$

where

$$\gamma=\frac{\gamma_0}{\vartheta_1 r_1+\vartheta_2 r_2}. \tag{251}$$

Substituting (249) into (246), we will find

$$p(\operatorname{arctg} x)=(1+x^2)g'(x). \tag{252}$$

Hence

$$g(a)-g(-a)=\int_{-a}^{a} g'(t)\,dt=\int_{-a}^{a}\frac{p(\operatorname{arctg} t)\,dt}{1+t^2},$$

or, assuming $t=\operatorname{tg}\varphi'$,

$$g(a)-g(-a)=\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'. \tag{253}$$

Substituting (240) into (251), we will find

$$\gamma=\frac{\theta_2 r_2}{\theta_1 r_1+\theta_2 r_2}\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'+\frac{r_2-r_1}{2(\theta_1 r_1+\theta_2 r_2)}. \tag{254}$$

Introducing designation

$$\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi'=q, \tag{255}$$

we will be able to give formula (254) the form

$$\gamma=\frac{2\theta_2 r_2 q+r_2-r_1}{2(\theta_1 r_1+\theta_2 r_2)}. \tag{256}$$

Substituting (252) in (250) and using designation (256), we will have

$$\int_{-a}^{a}\frac{g'(t)\,dt}{t-x}+\frac{\alpha_1 r_1+\alpha_2 r_2}{(\theta_1 r_1+\theta_2 r_2)(1+x^2)}g(x)=-\frac{qx+\gamma\operatorname{arctg} x}{1+x^2},\quad -a<x<a. \tag{257}$$

Assuming in (252) $x=\operatorname{tg}\varphi$, we will find

$$p(\varphi)=\sec^2\varphi\, g'(\operatorname{tg}\varphi). \tag{258}$$

Thus, detecting from equation (257) function $g(x)$, by the formula (258) we will find the unknown pressure $p(\phi)$. Equation (257) contains two unknown constants q and $a = \operatorname{tg} \phi_0$. Thus, the expression for pressure $p(\phi)$ found by formula (258) will contain unknown constants q and ϕ_0. Designating the resultant of the external compressing forces by P we will, as one can see from Fig. 28, have

$$r_1 \int_{-\varphi_0}^{\varphi_0} p(\varphi') \cos\varphi' \, d\varphi' = P. \tag{259}$$

Substituting the expression found for $p(\phi)$ into (256) and (259), we will obtain two equations from which we will find the unknown constants q and ϕ_0.

Thus, the contact problem examined in this section is reduced by us to the solution of equation (257). By introducing designations

$$\lambda(x) = \frac{\varkappa_1 r_1 + \varkappa_2 r_2}{\vartheta_1 r_1 + \vartheta_2 r_2} \frac{1}{1+x^2}, \tag{260}$$

$$f(x) = -\frac{\varepsilon x + \gamma \operatorname{arctg} x}{1+x^2}, \tag{261}$$

we will be able to write equation (257) in the form

$$\lambda(x) g(x) + \int_{-a}^{a} \frac{g(t)\,dt}{t-x} = f(x), \quad -a < x < a. \tag{262}$$

Equation (262) coincides with equation (411) in § 7 of Chapter I. As we showed in Chapter I, by replacing $\lambda(x)$ by the approximate expression of the form

$$\lambda(x) \approx \frac{a_0 + a_1 x + \ldots + a_n x^n}{b_0 + b_1 x + \ldots + b_n x^n} \sqrt{a^2 - x^2}, \tag{263}$$

we will obtain the equation for which it is possible to indicate the exact solution

$$g(x) = -\frac{1}{\pi^2}\int_0^x [\pi^2 F(t) + \lambda(t) R(t)] \cos[\mu(x) - \mu(t)]\, dt +$$

$$+ \frac{1}{\pi}\int_0^x \left[f(t) - \frac{g(a)}{a-t} - \frac{g(-a)}{a+t}\right] \sin[\mu(x) - \mu(t)]\, dt +$$

$$+ c_1 \cos\mu(x) + c_2 \sin\mu(x), \tag{264}$$

where

$$F(x) = -\frac{1}{\pi^2}\sqrt{a^2 - x^2}\int_{-a}^{a} \frac{f(t)\, dt}{\sqrt{a^2 - t^2}(t-x)}, \tag{265}$$

$$R(x) = \frac{\alpha_0 + \alpha_1 x + \ldots + \alpha_{n-1} x^{n-1}}{a_0 + a_1 x + \ldots + a_n x^n}, \tag{266}$$

$$\mu(x) = \frac{1}{\pi}\int_0^x \lambda(t)\, dt. \tag{267}$$

The method of determination of constants $c_1, c_2, \alpha_0, \alpha_1, \ldots, \alpha_{n-1}$, which enter into the indicated solution, was also shown in Chapter I.

Let us note that in our case

$$g(-a) = -g(a). \tag{268}$$

Indeed, in virtue of the symmetry function $p(\phi)$ should be even, and, consequently, function $g'(x)$, according to (252), should also be even. Consequently,

$$\int_{-a}^{a} \frac{g'(t)\, dt}{t} = 0. \tag{269}$$

Further, as one can see from (261),

$$f(0) = 0. \tag{270}$$

Assuming in (262) $x = 0$ and taking into account (269) and (270), we will find

$$g(0) = 0. \tag{271}$$

Thus

$$g(-a)=\int_{0}^{-a} g'(x)\,dx=-\int_{0}^{a} g'(x)\,dx=-g(a),$$

which was required to prove.

From relation (253), (255) and (268) it follows that

$$g(a)=\frac{q}{2},\quad g(-a)=-\frac{q}{2}. \tag{272}$$

Substituting (272) into (264), we will find

$$g(x)=-\frac{1}{\pi^3}\int_{0}^{x}[\pi^2F(t)+\lambda(t)R(t)]\cos[\mu(x)-\mu(t)]\,dt+\frac{1}{\pi}\int_{0}^{x}\Big[f(t)-$$
$$-\frac{qt}{a^2-t^2}\Big]\sin[\mu(x)-\mu(t)]\,dt+c_1\cos\mu(x)+c_2\sin\mu(x). \tag{273}$$

Further from formula (267) it follows that

$$\mu(0)=0. \tag{274}$$

Assuming in (273) $x = 0$ and taking into account (271) and (274), we will find

$$c_1=0. \tag{275}$$

Assuming in (273) $x = a$ and taking into account (275) and (272), we will find

$$c_2\cos\mu(a)=\frac{q}{2}+\frac{1}{\pi^3}\int_{0}^{a}[\pi^2F(t)+\lambda(t)R(t)]\cos[\mu(a)-\mu(t)]\,dt-$$
$$-\frac{1}{\pi}\int_{0}^{a}\Big[f(t)-\frac{qt}{a^2-t^2}\Big]\sin[\mu(a)-\mu(t)]\,dt; \tag{276}$$

Substituting (273) into equation (427) of § 7 of Chapter I and taking into account (275), we will obtain n equations

$$a_k = -\frac{1}{\pi^2}\int_{-a}^{a}\left\{\int_a^x [\pi^2 F(t)+\lambda(t)R(t)]\cos[\mu(x)-\mu(t)]\,dt - \right.$$

$$\left. -\pi\int_0^x\left[f(t)-\frac{qt}{a^2-t^2}\right]\sin[\mu(x)-\mu(t)]\,dt-\pi^2 c_2\sin\mu(x)\right\}\times$$

$$\times\frac{P_k(x)}{b_0+b_1x+\ldots+b_nx^n}\,dx,\quad k=0,\ 1,\ \ldots,\ n-1, \qquad (277)$$

where

$$P_k(x)=(b_0+b_1x+\ldots+b_kx^k)(a_{k+1}+a_{k+2}x+\ldots+a_nx^{n-k-1})-$$

$$-(a_0+a_1x+\ldots+a_kx^k)(b_{k+1}+b_{k+2}x+\ldots+b_nx^{n-k-1}),$$

$$k=0,\ 1,\ \ldots,\ n-1 \qquad (278)$$

according to the formulas (425) of § 7 of Chapter I.

Thus, we obtained the system of n + 1st equation (276) and (277) for the determination of constants $a_0, a_1, \ldots, a_{n-1}$, which enter into expression (266) of function $R(x)$, and constant c_2.

Differentiating (273) with respect to x and taking into account (267) and (275), we will find

$$g'(x)=-F(x)-\frac{\lambda(x)}{\pi^2}R(x)+\frac{\lambda(x)}{\pi^2}\int_a^x[\pi^2F(t)+\lambda(t)R(t)]\sin[\mu(x)-$$

$$-\mu(t)]\,dt+\frac{\lambda(x)}{\pi^2}\int_0^x\left[f(t)-\frac{qt}{a^2-t^2}\right]\cos[\mu(x)-\mu(t)]\,dt+$$

$$+\frac{c_2}{\pi}\lambda(x)\cos\mu(x). \qquad (279)$$

By introducing designation

$$\frac{1}{\pi}\frac{a_1r_1+a_2r_2}{b_1r_1+b_2r_2}=k, \qquad (280)$$

we will find according to (260) and (267)

$$\left.\begin{aligned}\lambda(x)&=\frac{ak}{1+x^2},\\ \mu(x)&=k\,\mathrm{arctg}\,x.\end{aligned}\right\} \qquad (281)$$

Substituting (279) into (258) and taking into account (281), we will obtain the following expression for the unknown function $p(\phi)$:

$$p(\varphi) = -\sec^2\varphi F(\operatorname{tg}\varphi) - \frac{k}{\pi} R(\operatorname{tg}\varphi) + \\ + \frac{k}{\pi^2} \int_0^{\operatorname{tg}\varphi} \left[\pi^2 F(t) + \frac{\pi k R(t)}{1+t^2}\right] \sin k(\varphi - \operatorname{arctg} t)\, dt + \\ + \frac{k}{\pi} \int_0^{\operatorname{tg}\varphi} \left[f(t) - \frac{q t}{a^2 - t^2}\right] \cos k(\varphi - \operatorname{arctg} t)\, dt + c_1 k \cos k\varphi. \quad (282)$$

Substituting (282) into (255) and (259), we will find constants q and $a = \operatorname{tg} \phi_0$, which enter into the expression obtained for $p(\phi)$, and we complete the solution of the contact problem examined in this section.

In conclusion of this section we will give graphs of the pressure[1] $p(\phi)$ for three values of angle ϕ_0, namely, for $\phi_0 = 30°$, $\phi_0 = 50°$ and $\phi_0 = 60°$ (Fig. 33), calculated by author for the case when the elastic constants of both compressible bodies are identical and Poisson's ratio is equal to 0.3. The dashed line on these graphs shows the distribution of pressure found with respect to known approximate formula[2]

$$p(\varphi) = \frac{P}{r_1 (\sin\varphi_0 \cos\varphi_0 + \varphi_0)} \cos\varphi,$$

which provides contact of the compressible bodies along the cylindrical surface of radius r_1 in the region of values of angle ϕ:

$$-\varphi_0 < \varphi < \varphi_0.$$

Figure 34 shows the dependence between angle ϕ_0 and applied force P (in Fig. 33 and Fig. 34 E – elastic modulus $\varepsilon = r_2 - r_1$). The dashed

[1]For the calculation of them see Appendix 2.

[2]Pathon, Ye. O. and Gorbunov, B. N. Steel bridges, Vol. II, Edition 3, Kiev, 1931, p. 23.

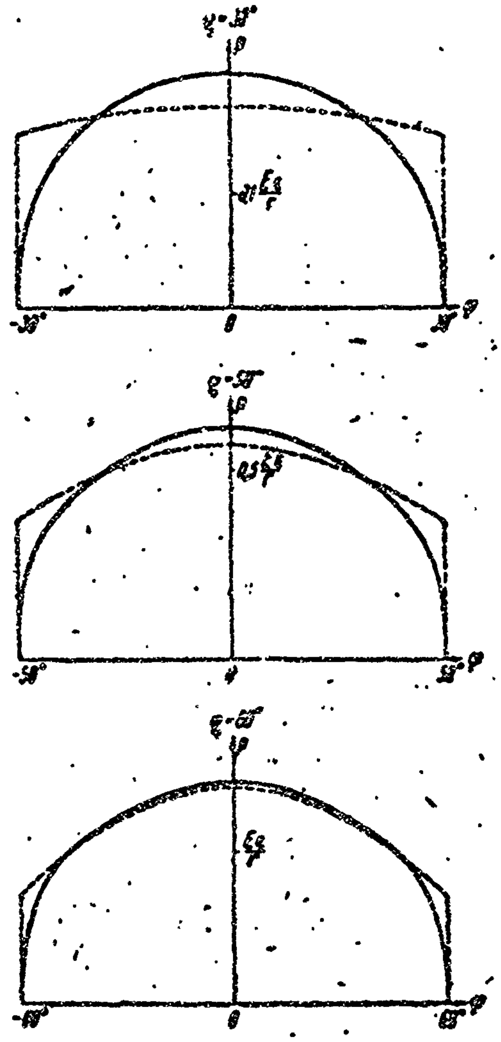

Fig. 33.

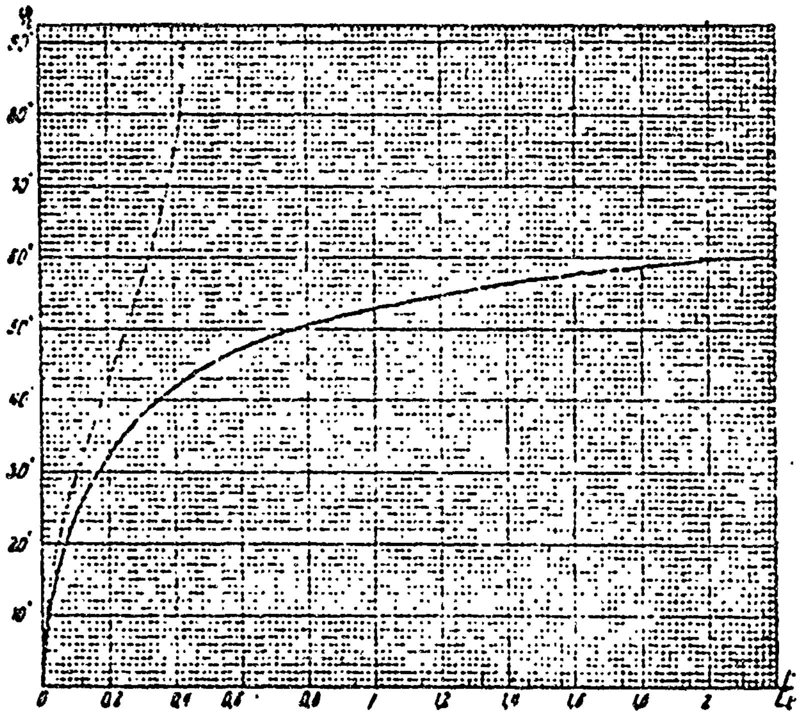

Fig. 34.

line shows the dependence between these values, which is calculated in accordance with the theory of Hertz discussed by us in § 1 of this chapter. This comparison clearly demonstrates unacceptability of the theory of Hertz in the contact problem examined by us in this section.

§ 8. Solution to the Problem about the Pressure of a Rigid Stamp on an Elastic Half-Plane on the Basis of a New Hypothesis

The problem about the compression of two elastic bodies was first solved on the assumption that both bodies have an ideally smooth surface. In a number of contemporary investigations the compression of elastic bodies in the presence of friction between them is examined. It would be more correct, meanwhile, in the solution of the contact problem to consider the microstructure of the surface of the compressible bodies. Contemporary physics does not give any completed theory of the surface structure of a solid. In view of this in this section we proceed in the solution of the contact problem from the following assumptions. If acting on the surface of an elastic body is a certain normal pressure, then this pressure causes elastic displacements in the body, which appear due to the deformation of the whole elastic body on the whole and which are determined by differential equations of the theory of elasticity. Thus, for example, if acting on the boundary of the elastic half-plane is the normal pressure $p(x)$ in region $-a<x<a$, then the point lying on the boundary of the elastic half-plane having the abscissa x accomplishes a normal elastic displacement equal to (see formula [illegible] of Chapter II)

$$\vartheta \int_{-a}^{a} p(t) \ln \frac{1}{|t-x|} dt + \text{const.}, \quad \vartheta = \frac{2}{\pi E}(1-\mu^2). \tag{283}$$

[illegible] ...ides the indicated displacements, the normal pressure [illegible] given point of the surface of the elastic body should cause [illegible] still a certain additional normal displacement, [illegible] to the purely local surface strains predetermined [illegible] ...ructure of the given elastic body. In this section

we will assume that this displacement is proportional to the normal pr ssure acting at a given point of the surface of the elastic body. Thus, if acting on the boundary of the elastic half-plane is the normal pressure $p(x)$ in region $-a < x < a$, then this additional displacement will be equal to zero when $|x| > a$ and will equal

$$kp(x) \tag{284}$$

when $|x| < a$, where k is a certain coefficient dependent on the surface structure of the elastic body. The complete normal displacement of the end point of the elastic half-space caused by the normal pressure $p(x)$ will be equal to the sum of displacements determined by formulas (283) and (284), i.e., will equal

$$kp(x) + \theta \int_{-a}^{a} p(t) \ln \frac{1}{|t-x|}\, dt + \text{const.}[1], \quad -a < x < a. \tag{285}$$

Let us now return to the problem about the pressure of a rigid stamp on the elastic half-plane examined by us in § 3 of this chapter. If the stamp has a flat base with width $2a$, then when $-a < x < a$ the normal displacement of end points of the elastic half-plane should remain constant. Thus, according to expression (285), for this normal displacement to determine the pressure $p(x)$ under the stamp we will have equation[2]

$$kp(x) + \theta \int_{-a}^{a} p(t) \ln \frac{1}{|t-x|}\, dt = \text{const.}, \quad -a < x < a \tag{286}$$

instead of equation (115).

Differentiating with respect to x both sides of equation (286), we will obtain equation

[1]Let us note that the hypothesis advanced by us in this section represents a unique combination of Hertz's theory with Winkler's hypothesis. When $k = 0$ we arrive at Hertz's theory, and when $\frac{k}{\theta} \to$ we arrive at Winkler's theory.

[2]See Appendix 2, Section 4, for a numerical solution of equation (286).

$$\lambda p'(x) + \int_{-a}^{a} \frac{p(t)\,dt}{t-x} = 0, \quad -a<x<a, \tag{287}$$

where $\lambda = \frac{k}{b}$.

This equation for function $p(x)$ coincides with equation (450) in § 7 of Chapter I for function $g(x)$ when

$$f(x) = 0, \quad \lambda(x) = \text{const}.$$

According to formula (467) in § 7 of Chapter I, we will find in this case

$$p(x) = \frac{1}{\pi}\int_{0}^{x} \left[S(t) + \frac{p(a)}{a-t} + \frac{p(-a)}{a+t}\right] \sin\frac{\pi}{\lambda}(x-t)\,dt + \\ + c_1\cos\frac{\pi x}{\lambda} + c_2\sin\frac{\pi x}{\lambda}, \tag{288}$$

where

$$S(x) = \frac{\beta_0+\beta_1 x+\dots+\beta_{n-1}x^{n-1}}{a_0+a_1x+\dots+a_nx^n}. \tag{289}$$

Having constructed for λ the approximate expression of the form

$$\lambda \approx \frac{a_0+a_1x+\dots+a_nx^n}{b_0+b_1x+\dots+b_nx^n}\sqrt{a^2-x^2}, \tag{290}$$

we will find constants $\beta_0, \beta_1, \dots, \beta_{n-1}$, which enter into expressions for $S(x)$, from equations

$$\int_{-a}^{a} p(t)\frac{d}{dt}\left[\frac{P_k(t)}{b_0+b_1t+\dots+b_nt^n}\right]dt = \frac{p(a)P_k(a)}{b_0+b_1a+\dots+b_na^n} - \\ - \frac{p(-a)P_k(-a)}{b_0-b_1a+\dots+b_n(-a)^n} - \beta_k, \quad k=0, 1, \dots, n-1, \tag{291}$$

where

$$P_k(t) = (b_0+b_1t+\dots+b_kt^k)(a_{k+1}+a_{k+2}t+\dots+a_nt^{n-k-1}) - \\ - (a_0+a_1t+\dots+a_kt^k)(b_{k+1}+b_{k+2}t+\dots+b_nt^{n-k-1}), \\ k=0, 1, \dots, n-1, \tag{292}$$

(see formulas (469) and (425) in § 7 of Chapter I).

With the proper selection of coefficients k, k_1, and k_2, function

$$\psi(x) = \frac{k}{a}\sqrt{a^2 - x^2}\,\frac{a^2 + k_1 x^2}{a^2 + k_2 x^2} \tag{293}$$

in the interval $-a < x < a$ is close to unity. Thus, for example, if, following N. I. Muskhelishvilli[1] we set

$$k = 1, \quad k_1 = 0{,}9, \quad k_2 = 0, \tag{294}$$

we will have the following values of function $\psi(x)$:

x	0,1 a	0,2 a	0,3 a	0,4 a	0,5 a	0,6 a	0,7 a	0,8 a	0,9 a
$\psi(x)$	1,00	1,02	1,03	1,05	1,06	1,06	1,03	0,95	0,75

Thus, it is possible approximately to assume that

$$\lambda \approx \lambda\psi(x) = \frac{\lambda k}{a}\,\frac{a^2 + k_1 x^2}{a^2 + k_2 x^2}\sqrt{a^2 - x^2}. \tag{295}$$

By comparing (295) with (290), we will find

$$n = 2, \; a_0 = \lambda k a^2, \; a_1 = 0, \; a_2 = \lambda k k_1, \quad b_0 = a^2, \; b_1 = 0, \; b_2 = a k_2. \tag{296}$$

Substituting (296) into (292), we will obtain

$$\left.\begin{array}{l} P_0(t) = a^2\lambda k k_1 t - \lambda k a^2 a k_2 t = \lambda k a^2 (k_1 - k_2)\, t, \\ P_1(t) = a^2\lambda k k_1 - \lambda k a^2 a k_2 = \lambda k a^2 (k_1 - k_2). \end{array}\right\} \tag{297}$$

Substituting (296) and (297) into (291), we will obtain equations

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[1]See N. I. Muskhelishvili, Singular integral equations, p. 386.

$$\left.\begin{aligned}\int_{-a}^{a} p(t)\frac{d}{dt}\left[\frac{t}{a^2+k_2t^2}\right]dt &= \frac{p(a)}{a(1+k_2)}+\frac{p(-a)}{a(1+k_2)}-\frac{\beta_0}{\lambda k a^2(k_1-k_2)},\\ \int_{-a}^{a} p(t)\frac{d}{dt}\left[\frac{1}{a^2+k_2t^2}\right]dt &= \frac{p(a)}{a^2(1+k_2)}-\frac{p(-a)}{a^2(1+k_2)}-\frac{\beta_1}{\lambda k a^2(k_1-k_2)}.\end{aligned}\right\} \tag{298}$$

But since in virtue of symmetry, function $p(x)$ should be even, then

$$\int_{-a}^{a} p(t)\frac{d}{dt}\left[\frac{1}{a^2+k_2t^2}\right]dt = -\int_{-a}^{a}\frac{2k_2tp(t)}{(a^2+k_2t^2)^2}dt=0, \tag{299}$$

$$p(-a)=p(a). \tag{300}$$

Substituting (299) and (300) into the second of equations (298), we will find

$$\beta_1=0. \tag{301}$$

and the first of equations (298) can be represented in the form

$$\int_{-a}^{a} p(t)\frac{a^2-k_2t^2}{(a^2+k_2t^2)^2}dt=\frac{2p(a)}{a(1+k_2)}-\frac{\beta_0}{\lambda k a^2(k_1-k_2)}. \tag{302}$$

Substituting (296) and (301) into (289), we will find

$$S(x)=\frac{\beta_0}{\lambda k(a^2+k_1x^2)}. \tag{303}$$

Substituting (303) and (300) into (288), we will obtain

$$p(x)=\frac{\beta_0}{\pi\lambda k}\int_0^x\frac{\sin\frac{\pi}{\lambda}(x-t)}{a^2+k_1t^2}dt+\frac{2ap(a)}{\pi}\int_0^x\frac{\sin\frac{\pi}{\lambda}(x-t)}{a^2-t^2}dt+$$

$$+c_1\cos\frac{\pi x}{\lambda}+c_2\sin\frac{\pi x}{\lambda}. \tag{304}$$

Assuming in (304)

$$x=a\xi,\quad t=a\tau, \tag{305}$$

we will find

$$p(a\xi) = \frac{\beta_0}{\pi\lambda k a}\int_0^{\xi}\frac{\sin\frac{\pi a}{\lambda}(\xi-\tau)\,d\tau}{1+k_1\tau^2} + \frac{2}{\pi}\,p(a)\int_0^{\xi}\frac{\sin\frac{\pi a}{\lambda}(\xi-\tau)\,d\tau}{1-\tau^2} +$$

$$+ c_1\cos\frac{\pi a\xi}{\lambda} + c_2\sin\frac{\pi a\xi}{\lambda}. \tag{306}$$

Let us introduce further the designations

$$\frac{\beta_0}{2\lambda k a} = \beta, \quad \frac{\pi a}{\lambda} = c, \tag{307}$$

$$f_1(\xi, c) = \frac{2}{\pi}\int_0^{\xi}\frac{\sin c(\xi-\tau)\,d\tau}{1+k_1\tau^2}, \quad f_2(\xi, c) = \frac{2}{\pi}\int_0^{\xi}\frac{\sin c(\xi-\tau)\,d\tau}{1-\tau^2}. \tag{308}$$

Then formula (306) can be given the form

$$p(a\xi) = \beta f_1(\xi, c) + p(a)\,f_2(\xi, c) + c_1\cos c\xi + c_2\sin c\xi. \tag{309}$$

Since function $p(x)$ should be even, we should have

$$c_2 = 0. \tag{310}$$

Assuming in (309) $\xi = 1$ and taking into account (310), we will find

$$p(a) = \beta f_1(1, c) + p(a)\,f_2(1, c) + c_1\cos c. \tag{311}$$

Substituting into (302) expressions for t and β_0 from (305) and (307), we will obtain

$$\frac{1}{2}\int_{-1}^{1} p(a\tau)\,\frac{1-k_1\tau^2}{(1+k_1\tau^2)^2}\,d\tau = \frac{p(a)}{1+k^2} - \frac{\beta}{k_1-k_2}. \tag{312}$$

If further we designate compressing force by P, we will have

$$\int_{-a}^{a} p(t)\,dt = P,$$

or, if one were to assume $t = a\tau$:

$$\int_{-1}^{1} p(a\tau)\,d\tau = \frac{P}{a}. \tag{313}$$

Substituting (310) into (309), we will find

$$p(a\xi) = \beta f_1(\xi, c) + p(a) f_2(\xi, c) + c_1 \cos c\xi. \tag{314}$$

Substituting (314) into (312) and (313) and joining relation (311) to the obtained equations, we will obtain a system of three linear equations:

$$\left.\begin{aligned}
&\beta\left[\frac{1}{2}\int_{-1}^{1} f_1(\tau, c)\,\frac{1-k_2\tau^2}{(1+k_2\tau^2)^2}\,d\tau + \frac{1}{k_1-k_2}\right] + \\
&+ p(a)\left[\frac{1}{2}\int_{-1}^{1} f_2(\tau, c)\,\frac{1-k_2\tau^2}{(1+k_2\tau^2)^2}\,d\tau - \frac{1}{1+k_2}\right] + \\
&\qquad\qquad + \frac{c_1}{2}\int_{-1}^{1} \cos c\tau\,\frac{1-k_2\tau^2}{(1+k_2\tau^2)^2}\,d\tau = 0, \\
&\beta\int_{-1}^{1} f_1(\tau, c)\,d\tau + p(a)\int_{-1}^{1} f_2(\tau, c)\,d\tau + 2c_1\,\frac{\sin c}{c} = \frac{P}{a}, \\
&\qquad \beta f_1(1, c) + p(a)\,[f_2(1, c) - 1] + c_1 \cos c = 0,
\end{aligned}\right\} \tag{315}$$

for the determination of three unknown constants β, $p(a)$ and c_1. Having detected β, $p(a)$ and c_1 from equations (315) and substituting their values into (314), we will find the unknown pressure under the stamp $p(x)$.

Figure 35 shows graphs of pressure $p(x)$ under the stamp for the following values of parameter $c = \frac{\gamma a}{k}$:

$$c = 0,\ c = 0.1,\ c = 1,$$
$$c = 10 \text{ and } c = \infty.$$

Case $c = \infty$ corresponds to the conventional theory of the stamp, in accordance with which when $|x| = a$ the pressure $p(x)$ turns into infinity.

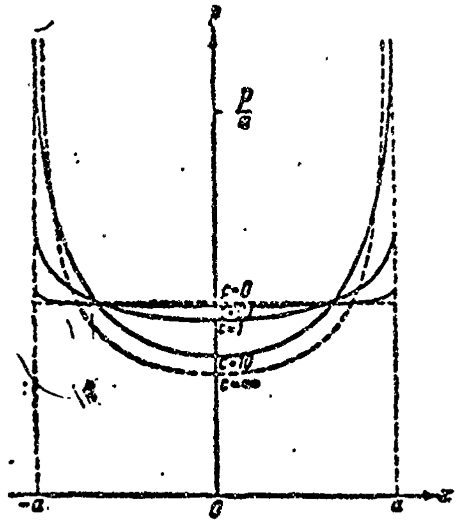

Fig. 35.

At finite values of the parameter c pressure $p(x)$ remains limited, and in the limiting case $c = 0$ we will obtain the evenly distributed pressure under the stamp.

C H A P T E R III

AXISYMMETRICAL CONTACT PROBLEM

§ 1. Mathematical Introduction to the Axisymmetrical Contact Problem of the Theory of Elasticity

In this section we show the general solution of the fundamental equation of the axisymmetrical contact problem of the theory of elasticity

$$\iint\limits_{\Sigma} \frac{p(r')\,d\sigma}{R} = f(r), \quad 0 < r < a, \tag{1}$$

where region of integration Σ is a circle of radius a (Fig. 36), $d\sigma$ – element of the area of this circle, R – distance between point A, which remains motionless in the process of integration, and point A' included inside element $d\sigma$, $p(r)$ – unknown function of distance r' from point A' to the center of circle O subject to determination from equation (1), and, finally, $f(r)$ – assigned function of distance r from point A to the center of circle O. With respect to the assigned function $f(r)$, below we will assume that it is continuous together with its first derivative when $0 < r < a$.

Let us present integral extended on the area of circle Σ, which stands in the left side of equation (1) in the form of a multiple integral, selecting polar coordinates R and ψ as variables of integration with the center at point A. Integration with respect to the variable R must be produced within 0 to R_0, where R_0 – distance

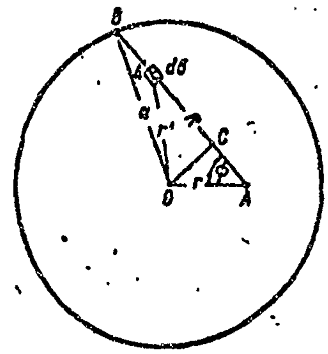

Fig. 36.

between points A and B in Fig. 36, and subsequent integration with respect to ψ – within 0 to 2π. The area of element $d\sigma$ is equal to $RdRd\psi$ and, thus, equation (1) will take the form

$$\int_0^{2\pi} d\psi \int_0^{R_0} p(r')\, dR = f(r), \quad 0 < r < a. \tag{2}$$

Distances r' and R_0 entering into equation (2) are expressed in terms of variables R and ψ by relations (see Fig. 36)

$$r' = \sqrt{R^2 - 2Rr\cos\psi + r^2}, \tag{3}$$

$$R_0 = r\cos\psi + \sqrt{a^2 - r^2\sin^2\psi}. \tag{4}$$

Let us introduce into the consideration angle ϑ, determined by relation

$$\cos\vartheta = \frac{R - r\cos\psi}{\sqrt{a^2 - r^2\sin^2\psi}}, \quad 0 \leqslant \vartheta \leqslant \pi \tag{5}$$

(as can be seen from Fig. 36, the ratio standing in the right side of equality (5), in absolute value does not exceed unity). Let us designate further by $\theta(\psi)$ that value which takes angle ϑ when $R = 0$:

$$\cos\theta(\psi) = -\frac{r\cos\psi}{\sqrt{a^2 - r^2\sin^2\psi}}, \tag{6}$$

and let us turn in equation (2) from the variable of integration R to the variable ϑ. From (5) we find

$$R = r\cos\psi + \cos\vartheta\sqrt{a^2 - r^2\sin^2\psi},$$
$$dR = -\sin\vartheta\sqrt{a^2 - r^2\sin^2\psi}\,d\vartheta,$$

whence

$$\int_0^{R_0} p(r')\,dR = \int_0^{\theta(\psi)} p(r')\sqrt{a^2 - r^2\sin^2\psi}\,\sin\vartheta\,d\vartheta, \qquad (7)$$

since according to relations (4), (5) and (6) angle ϑ is changed from 0 to $\theta(\psi)$ when R is changed from R_0 to 0. Substituting (7) into (2), we obtain equation

$$\int_0^{2\pi} d\psi \int_0^{\theta(\psi)} p(r')\sqrt{a^2 - r^2\sin^2\psi}\,\sin\vartheta\,d\vartheta = f(r), \qquad 0 < r < a. \qquad (8)$$

From relations (3) and (5) we find

$$\begin{aligned} r' &= \sqrt{(R - r\cos\psi)^2 + r^2\sin^2\psi} = \sqrt{\cos^2\vartheta\,(a^2 - r^2\sin^2\psi) + r^2\sin^2\psi} = \\ &= \sqrt{(a^2 - r^2\sin^2\psi)(1 - \sin^2\vartheta) + r^2\sin^2\psi} = \\ &= \sqrt{a^2 - (a^2 - r^2\sin^2\psi)\sin^2\vartheta}. \end{aligned} \qquad (9)$$

Equation (8) can be presented in the form

$$\begin{aligned} &\int_0^{\pi} d\psi \int_0^{\theta(\psi)} p(r')\sqrt{a^2 - r^2\sin^2\psi}\,\sin\vartheta\,d\vartheta + \\ &+ \int_{\pi}^{2\pi} d\psi \int_0^{\theta(\psi)} p(r')\sqrt{a^2 - r^2\sin^2\psi}\,\sin\vartheta\,d\vartheta = f(r), \quad 0 < r < a. \end{aligned} \qquad (10)$$

Replacing ϑ by $\pi - \vartheta$, we find

$$\int_0^{\theta(\psi)} p(r')\sin\vartheta\,d\vartheta = \int_{\pi-\theta(\psi)}^{\pi} p(r')\sin\vartheta\,d\vartheta, \qquad (11)$$

since according to (9) with this replacement r' retains its value. Using the relation (11) and replacing ψ by $\pi + \psi$, we obtain

$$\int_{\pi}^{2\pi} d\psi \int_{0}^{\theta(\psi)} p(r')\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta =$$

$$= \int_{\pi}^{2\pi} d\psi \int_{\pi-\theta(\psi)}^{\pi} p(r')\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\psi \int_{\pi-\theta(\pi+\psi)}^{\pi} p(r')\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta. \qquad (12)$$

But from relation (6) it follows that

$$\cos\theta(\pi+\psi) = \frac{r\cos\psi}{\sqrt{a^2 - r^2 \sin^2\psi}} = -\cos\theta(\psi),$$

whence

$$\theta(\pi+\psi) = \pi - \theta(\psi), \qquad (13)$$

since according to condition $0 < \theta(\psi) < \pi$. Substituting (13) into (12), we find

$$\int_{\pi}^{2\pi} d\psi \int_{0}^{\theta(\psi)} p(r')\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\psi \int_{\theta(\psi)}^{\pi} p(r')\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta. \qquad (14)$$

Substituting (14) into (10), we obtain equation

$$\int_{0} d\psi \int_{\theta}^{\pi} p(r')\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta = f(r), \quad 0 < r < a. \qquad (15)$$

Substituting (9) into (15), we will have

$$\int_{0}^{\pi} d\psi \int_{0}^{\pi} p\left[\sqrt{a^2 - (a^2 - r^2 \sin^2\psi)\sin^2\theta}\right]\sqrt{a^2 - r^2 \sin^2\psi}\,\sin\theta\, d\theta = f(r), \qquad (16)$$

$$0 < r < a.$$

Let us introduce further designation

$$F(r)=2\int_0^{\frac{\pi}{2}} p[\sqrt{a^2-(a^2-r^2)\sin^2\theta}]\sqrt{a^2-r^2}\sin\theta\, d\theta,\ 0<r<a. \tag{17}$$

Using this designation, we can give to equation (16) the form

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} F(r\sin\psi)\,d\psi=f(r), \quad 0<r<a. \tag{18}$$

Setting in (17) $r=\sqrt{a^2-\rho^2}$, we obtain

$$\int_0^{\frac{\pi}{2}} p(\sqrt{a^2-\rho^2\sin^2\theta})\,\rho\sin\theta\, d\theta=\frac{1}{2}F(\sqrt{a^2-\rho^2}), \quad 0<\rho<a. \tag{19}$$

Introducing designations

$$\rho p(\sqrt{a^2-\rho^2})=g(\rho), \tag{20}$$

$$F(\sqrt{a^2-\rho^2})=4G(\rho), \tag{21}$$

we will be able to give to equation (19) the form

$$\frac{1}{2}\int_0^{\frac{\pi}{2}} g(\rho\sin\theta)\,d\theta=G(\rho), \quad 0<\rho<a. \tag{22}$$

Setting in (20) $\rho=\sqrt{a^2-r^2}$, we find:

$$p(r)=\frac{g(\sqrt{a^2-r^2})}{\sqrt{a^2-r^2}}, \quad 0<r<a. \tag{23}$$

From (21) it follows that

$$G(\rho)=\frac{1}{4}F(\sqrt{a^2-\rho^2}), \quad 0<\rho<a. \tag{24}$$

Thus, determining from equation (18) function $F(r)$, from formula (24) we will find function $G(\rho)$. Determining further from equation (22) function $g(\rho)$, by formula (23) we will find function $p(r)$, which represents the solution of the initial equation (1). Thus, we reduced the solution of equation (1) to the sequential solution of two identical equations (18) and (22) for functions $F(r)$ and $g(\rho)$.

Replacing ψ by $\pi-\psi$, we find

$$\int_{\pi/2}^{\pi} F(r\sin\psi)\,d\psi = \int_{0}^{\pi/2} F(r\sin\psi)\,d\psi,$$

whence

$$\int_{0}^{\pi} F(r\sin\psi)\,d\psi = \int_{0}^{\pi/2} F(r\sin\psi)\,d\psi + \int_{\pi/2}^{\pi} F(r\sin\psi)\,d\psi = 2\int_{0}^{\pi/2} F(r\sin\psi)\,d\psi. \tag{25}$$

Substituting (25) into (18), let us give to equation (18) the form:

$$\int_{0}^{\pi/2} F(r\sin\psi)\,d\psi = f(r), \quad 0<r<a, \tag{26}$$

similarly, equation (22) can be given the form

$$\int_{0}^{\pi/2} g(\rho\sin\theta)\,d\theta = G(\rho), \quad 0<\rho<a. \tag{27}$$

In order to obtain the solution to equations (26) and (27), let us prove that for any function $f(r)$ continuous together with its first derivative when $0<r<a$, there is the identity

$$\int_{0}^{\pi/2} d\varphi \int_{0}^{\pi/2} f'(r\sin\varphi\sin\psi)\,r\sin\psi\,d\psi = \frac{\pi}{2}[f(r)-f(0)], \quad 0<r<a. \tag{28}$$

Let us introduce instead of variable of integration ψ a new variable ξ, assuming

$$\xi = r\sin\psi.$$

Let us find

$$d\xi = r\cos\psi\,d\psi, \quad d\psi = \frac{d\xi}{\sqrt{r^2-r^2\sin^2\psi}} = \frac{d\xi}{\sqrt{r^2-\xi^2}},$$

$$\int_{0}^{\pi/2} f'(r\sin\varphi\sin\psi)\,r\sin\psi\,d\psi = \int_{0}^{r} f'(\xi\sin\varphi)\,\frac{\xi\,d\xi}{\sqrt{r^2-\xi^2}}. \tag{29}$$

Using relation (29) and changing the order of integration, we will find

$$\int_0^{\pi/2} d\varphi \int_0^{\pi/2} f'(r\sin\varphi\sin\psi)\, r\sin\psi\, d\psi = \\ = \int_0^{\pi/2} d\varphi \int_0^{r} f'(\xi\sin\varphi)\frac{\xi\, d\xi}{\sqrt{r^2-\xi^2}} = \int_0^{r} d\xi \int_0^{\pi/2} \frac{\xi f'(\xi\sin\varphi)}{\sqrt{r^2-\xi^2}}\, d\varphi. \tag{30}$$

Let us introduce further, instead of the variable of integration ϕ, the new variable η, assuming $\eta=\xi\sin\varphi$.

We will find

$$d\eta=\xi\cos\varphi\, d\varphi, \quad d\varphi=\frac{d\eta}{\sqrt{\xi^2-\xi^2\sin^2\varphi}}=\frac{d\eta}{\sqrt{\xi^2-\eta^2}}, \\ \int_0^{\pi/2} f'(\xi\sin\varphi)\, d\varphi=\int_0^{\xi} f'(\eta)\frac{d\eta}{\sqrt{\xi^2-\eta^2}}. \tag{31}$$

Substituting (31) into (30), we find

$$\int_0^{\pi/2} d\varphi \int_0^{\pi/2} f'(r\sin\varphi\sin\psi)\, r\sin\psi\, d\psi=\int_0^{r} d\xi \int_0^{\xi} \frac{\xi f'(\eta)}{\sqrt{(r^2-\xi^2)(\xi^2-\eta^2)}}\, d\eta. \tag{32}$$

If in the multiple integral standing in the right side of relation (32) we examine variables ξ and η as rectangular coordinates, then the region of integration will be the triangle shaded in Fig. 37. Actually, first at fixed ξ integration is conducted with respect to η from $\eta = 0$ to $\eta = \xi$, then integration with respect to ξ from 0 to r is produced. If in this multiple integral we change the order of integration, then in order that the region of integration is preserved, we must initially, with fixed η, integrate with respect to ξ from $\xi = \eta$ to $\xi = r$ and then integrate with respect to η from 0 to r (Fig. 37). Thus, if one were to change the order of integration, relation (32) will take the form

$$\int_0^{\pi/2} d\varphi \int_0^{\pi/2} f'(r\sin\varphi\sin\psi)\, r\sin\psi\, d\varphi=\int_0^{r} d\eta \int_{\eta}^{r} \frac{\xi f'(\eta)}{\sqrt{(r^2-\xi^2)(\xi^2-\eta^2)}}\, d\xi. \tag{33}$$

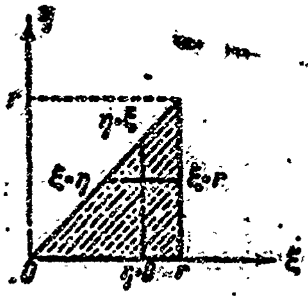

Fig. 37.

Let us replace now the variable of integration ξ by a new variable t, setting

$$t=\frac{2\xi^2-\eta^2-r^2}{r^2-\eta^2}.$$

Let us find

$$\left.\begin{array}{c}
dt=\dfrac{4\xi d\xi}{r^2-\eta^2}, \quad 1+t=\dfrac{2(\xi^2-\eta^2)}{r^2-\eta^2}, \quad 1-t=\dfrac{2(r^2-\xi^2)}{r^2-\eta^2}, \\
\dfrac{dt}{\sqrt{1-t^2}}=\dfrac{2\xi d\xi}{\sqrt{(r^2-\xi^2)(\xi^2-\eta^2)}}, \\
\displaystyle\int_\eta^r \frac{\xi d\xi}{\sqrt{(r^2-\xi^2)(\xi^2-\eta^2)}}=\frac{1}{2}\int_{-1}^{1}\frac{dt}{\sqrt{1-t^2}}=\frac{1}{2}\arcsin t\Big|_{-1}^{1}=\frac{\pi}{2}.
\end{array}\right\} \qquad (34)$$

Substituting (34) into (33), we obtain

$$\int_0^{\pi/2} d\varphi \int_0^{\pi/2} f'(r\sin\varphi\sin\psi)\, r\sin\varphi\, d\psi=\frac{\pi}{2}\int_0^r f'(\eta)\,d\eta=\frac{\pi}{2}[f(r)-f(0)],$$

QED.

Using identity (28), it is easy to obtain the solution to equation (26). Let us assume first that there exists function $F(r)$, which is continuous together with its first derivative when $0<r<a$ and which satisfies equation (26). Differentiating both sides of equation (26) with respect to r, we will obtain

$$\int_0^{\pi/2} F'(r\sin\psi)\sin\psi\, d\psi=f'(r), \quad 0<r<a. \qquad (35)$$

Replacing in (35) r by $r \sin \phi$ and integrating both sides of the obtained relation with respect to ϕ within limits from 0 to $\pi/2$, we will find

$$\int_0^{\pi/2} d\varphi \int_0^{\pi/2} F'(r\sin\varphi\sin\psi)\sin\psi\, d\psi = \int_0^{\pi/2} f'(r\sin\varphi)\, d\varphi, \quad 0<r<a. \qquad (36)$$

Multiplying both sides of relation (36) by r and taking into account identity (28), we obtain

$$\frac{\pi}{2}[F(r)-F(0)] = r\int_0^{\pi/2} f'(r\sin\varphi)\, d\varphi, \quad 0<r<a. \qquad (37)$$

Assuming in (26) $r = 0$, we find

$$\frac{\pi}{2}F(0) = f(0). \qquad (38)$$

Substituting (33) in (37), we will come to the conclusion that if equation (26) has a solution continuous together with its first derivative when $0<r<a$, then this solution should have the form

$$F(r) = \frac{2}{\pi}\left[f(0) + r\int_0^{\pi/2} f'(r\sin\varphi)\, d\varphi\right], \quad 0<r<a. \qquad (39)$$

Let us prove now that function $F(r)$, determined by formula (9), indeed satisfies equation (26). Using identity (28), let us find from (39)

$$\begin{aligned}
&\int_0^{\pi/2} F(r\sin\psi)\, d\psi = \\
&= \frac{2}{\pi}\left[f(0)\int_0^{\pi/2} d\psi + \int_0^{\pi/2} d\psi \int_0^{\pi/2} f'(r\sin\varphi\sin\psi)\, r\sin\psi\, d\varphi\right] = \\
&= \frac{2}{\pi}\left\{\frac{\pi}{2} f(0) + \frac{\pi}{2}[f(r)-f(0)]\right\} = f(r), \quad 0<r<a,
\end{aligned}$$

i.e., function $F(r)$, determined by formula (39), indeed satisfies equation (26).

Analogously, equation (27) will have the solution

$$g(\rho)=\frac{2}{\pi}\left[G(0)+\rho\int_0^{\pi/2}G'(\rho\sin\varphi)\,d\varphi\right],\quad 0<\rho<a. \tag{40}$$

Substituting (39) and (40) into (24) and (23), we will find

$$G(\rho)=\frac{1}{2\pi}\left[f(0)+\sqrt{a^2-\rho^2}\int_0^{\pi/2}f'\left(\sqrt{a^2-\rho^2}\sin\varphi\right)d\varphi\right],\quad 0<\rho<a, \tag{41}$$

$$p(r)=\frac{2}{\pi}\left[\frac{G(0)}{\sqrt{a^2-r^2}}+\int_0^{\pi/2}G'\left(\sqrt{a^2-r^2}\sin\varphi\right)d\varphi\right],\quad 0<r<a. \tag{42}$$

Formulas (41) and (42) determine the solution of the initial equation (1). It is possible, however, to present the solution in a more convenient form for calculations. From (24) we find

$$G(0)=\frac{1}{4}F(a),\quad G'(\rho)=-\frac{1}{4}F'\left(\sqrt{a^2-\rho^2}\right)\frac{\rho}{\sqrt{a^2-\rho^2}}. \tag{43}$$

Substituting (43) into (42), we obtain

$$p(r)=\frac{1}{2\pi}\left\{\frac{F(a)}{\sqrt{a^2-r^2}}-\int_0^{\pi/2}F'\left[\sqrt{a^2-(a^2-r^2)\sin^2\varphi}\right]\frac{\sqrt{a^2-r^2}\sin\varphi\,d\varphi}{\sqrt{a^2-(a^2-r^2)\sin^2\varphi}}\right\}. \tag{44}$$

Let us produce now in (44) replacement of the variable of integration ϕ, assuming

$$a^2-(a^2-r^2)\sin^2\varphi=s^2.$$

Let us find

$$(a^2-r^2)\sin\varphi\cos\varphi\,d\varphi=-s\,ds,$$

$$\sqrt{a^2-r^2}\sin\varphi\,d\varphi=-\frac{s\,ds}{\sqrt{a^2-r^2}\cos\varphi}=$$

$$=-\frac{s\,ds}{\sqrt{a^2-r^2-(a^2-r^2)\sin^2\varphi}}=-\frac{s\,ds}{\sqrt{s^2-r^2}},$$

$$\int_0^{\pi/2}F'\left[\sqrt{a^2-(a^2-r^2)\sin^2\varphi}\right]\frac{\sqrt{a^2-r^2}\sin\varphi\,d\varphi}{\sqrt{a^2-(a^2-r^2)\sin^2\varphi}}=\int_r^a\frac{F'(s)\,ds}{\sqrt{s^2-r^2}}. \qquad (45)$$

Substituting (45) in (44), we find

$$p(r)=\frac{1}{2\pi}\left[\frac{F(a)}{\sqrt{a^2-r^2}}-\int_r^a\frac{F'(s)\,ds}{\sqrt{s^2-r^2}}\right],\quad 0<r<a. \qquad (46)$$

Formula (46), jointly with formula (39), for function $F(r)$ gives the solution to the initial equation (1). Formula (39) can be given a form similar to the form of formula (46), if one were to replace the variable of integration ϕ by the new variable σ, assuming

$$r\sin\varphi=\sigma.$$

Let us find

$$d\varphi=\frac{d\sigma}{r\cos\varphi}=\frac{d\sigma}{\sqrt{r^2-\sigma^2}},\quad \int_0^{\pi/2}f'(r\sin\varphi)\,d\varphi=\int_0^r\frac{f'(\sigma)\,d\sigma}{\sqrt{r^2-\sigma^2}}. \qquad (47)$$

Substituting (47) into (39), we find

$$F(r)=\frac{2}{\pi}\left[f(0)+r\int_0^r\frac{f'(\sigma)\,d\sigma}{\sqrt{r^2-\sigma^2}}\right],\quad 0<r<a. \qquad (48)$$

In conclusion of this paragraph let us show one more simple formula for the solution of the initial equation (1).

Below we will assume that function $F(r)$ has a continuous second derivative. Differentiating both sides of relation (39) with respect to r, we will find

$$F'(r)=\frac{2}{\pi}\left[\int_0^{\pi/2} f'(r\sin\varphi)\,d\varphi+\int_0^{\pi/2} f''(r\sin\varphi)\,r\sin\varphi\,d\varphi\right], \tag{49}$$
$$0<r<a.$$

Assuming in (49) $r \sin \phi = \sigma$, we will have (see (47)

$$F'(r)=\frac{2}{\pi}\int_0^r \frac{f'(\sigma)+\sigma f''(\sigma)}{\sqrt{r^2-\sigma^2}}\,d\sigma, \quad 0<r<a. \tag{50}$$

Substituting (50) into (46), we find

$$p(r)=-\frac{1}{\pi^2}\int_r^a ds\int_0^s \frac{f'(\sigma)+\sigma f''(\sigma)}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}}\,d\sigma+\frac{c}{\sqrt{a^2-r^2}}, \quad 0<r<a, \tag{51}$$

where

$$c=\frac{1}{2\pi}F(a). \tag{52}$$

Substituting (48) into (52), we obtain

$$c=\frac{1}{\pi^2}\left[f(0)+a\int_0^a \frac{f'(\sigma)\,d\sigma}{\sqrt{a^2-\sigma^2}}\right]. \tag{53}$$

If in the multiple integral, which appears in formula (51), variables of integration s and σ are examined as rectangular coordinates, then the region of integration will be the trapezium shaded in Fig. 38. Actually, first at fixed s we integrate with respect to σ within $\sigma = 0$ to $\sigma = s$, and then we integrate with respect to s within r to a. If one were to change the order of integration, the initial integration with respect to s will have to be conducted within $s = r$ to $s = a$, if $\sigma < r$, and within $s = \sigma$ to $s = a$, if $\sigma > r$, i.e., within s_0 to a, where

$$\left.\begin{aligned} s_0&=r \text{ when } \sigma<r,\\ s_0&=\sigma \text{ when } \sigma>r. \end{aligned}\right\} \tag{54}$$

Subsequent integration with respect to σ should be produced within 0 to a. Thus, after a change of the order of integration

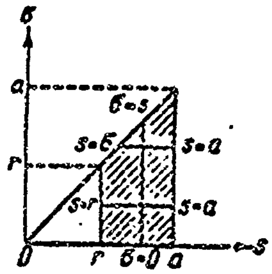

Fig. 38.

formula (51) will take the form

$$p(r) = -\frac{1}{\pi^2}\int_0^a d\sigma \int_{s_0}^{a} \frac{f'(s)+sf''(s)}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}}\,ds + \frac{c}{\sqrt{a^2-r^2}}, \quad 0<r<a. \tag{55}$$

When $\sigma < r$, assuming

$$s = \frac{r}{t}, \quad \sigma = kr, \tag{56}$$

we find according to (54)

$$\int_{s_0}^{a} \frac{ds}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}} = -\int_1^{r/a} \frac{r\,dt}{t^2r^2\sqrt{\left(\frac{1}{t^2}-1\right)\left(\frac{1}{t^2}-k^2\right)}} = \frac{1}{r}\int_{r/a}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{57}$$

The definite integral

$$F(x, k) = \int_0^x \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}} \tag{58}$$

is called the elliptic integral of the first kind, and there are detailed tables which give its value depending upon the upper limit x and parameter k, called the modulus elliptic integral. Thus, formula (57) can be given the form

$$\text{when } \sigma<r \quad \int_{s_0}^{a} \frac{ds}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}} = \frac{1}{r}\,[F(1, k) - F(x, k)], \quad x = \frac{r}{a}, \quad k = \frac{\sigma}{r} \tag{59}$$

according to (56).

When $\sigma > r$, assuming

$$s = \frac{\sigma}{t}, \quad r = \sigma k, \tag{60}$$

we will find according to (54)

$$\int_{s_0}^{a} \frac{ds}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}} = -\int_{1}^{\frac{\sigma}{a}} \frac{\sigma\, dt}{t^2\sigma^2 \sqrt{\left(\frac{1}{t^2}-k^2\right)\left(\frac{1}{t^2}-1\right)}} =$$
$$= \frac{1}{\sigma}\int_{\frac{\sigma}{a}}^{1} \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}. \tag{61}$$

Using designation (58), we will be able to give to formula (61) the form

$$\text{when } \sigma > r \quad \int_{s_0}^{a} \frac{ds}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}} = \frac{1}{\sigma}[F(1, k) - F(x, k)],$$
$$x = \frac{\sigma}{a}, \quad k = \frac{r}{\sigma} \tag{62}$$

according to (60).

Formulas (59) and (62) can be united into one:

$$\int_{s_0}^{a} \frac{ds}{\sqrt{(s^2-r^2)(s^2-\sigma^2)}} = \frac{1}{ax}[F(1, k) - F(x, k)], \tag{63}$$

where

$$\left.\begin{aligned} x &= \frac{r}{a}, \quad k = \frac{\sigma}{r} \text{ when } \sigma < r, \\ x &= \frac{\sigma}{a}, \quad k = \frac{r}{\sigma} \text{ when } \sigma > r. \end{aligned}\right\} \tag{64}$$

Substituting (63) into (55), we find

$$p(r) = -\frac{1}{\pi^2}\int_0^a \frac{f'(\sigma) + \sigma f''(\sigma)}{ax}[F(1, k) - F(x, k)]\, d\sigma + \frac{c}{\sqrt{a^2-r^2}},$$
$$0 < r < a. \tag{65}$$

In formula (65) x and k are functions of the variable of integration σ, which are determined by formula (64), and the constant c is determined by formula (53).

Thus, for the solution of equation (1) we obtained two formulas: formula (46), where function $F(r)$ is determined by relation (48), or relation (39), and formula (65). In those cases wnen function $F(r)$, appearing in formula (46), is determined in the elementary functions, it is advantageous to use formula (46) as being simpler. If, however, the definite integral, which appears in formula (48) for function $F(r)$, is not expressed in elementary functions and requires methods of approximation of the calculation, it is more profitable to use formula (65). Calculation of the integrand in formula (65) is easily carried out with the help of tables for elliptic integrals; after the series of values σ, included in the interval of integration (0, a) its numerical values are calculated, it is possible to calculate the value of the definite integral appearing in formula (65) according to any method of approximation from the theory of mechanical quadratures (by the trapezium formula, by the Simpson formula, etc.).

§ 2. Compression of Elastic Bodies Limited by Surfaces of Rotation

In this section we examine the axisymmetrical contact problem of the theory of elasticity, i.e., the problem on the compression of two elastic bodies limited by surfaces of rotation, where it is assumed that the axes of symmetry of the compressible bodies coincide and resultants of compressing forces lie on this general axis of symmetry. Let us construct the system of cylindrical coordinates, r, ϕ, z, directing the z axis along the common axis of symmetry of compressible bodies and disposing origin of the coordinates at the point of contact of the elastic bodies (Fig. 39). Let us assume that

$$z=z_1(r) \text{ and } z=-z_2(r) \tag{66}$$

equations of surfaces limiting the compressible bodies (we will consider as the first - that body inside which positive semiaxis z passes). Let us assume that further A_1 and A_2 are two points of

Fig. 39.

surfaces of compressible bodies arriving in contact with compression, and u_1 and u_2 are their elastic displacements. The distance a betwee points B_1 and B_2 in Fig. 39 constitutes the approach of elastic bodies with compression and will be constant for no matter how many pairs of points arriving in contact with compression we fulfilled the construction shown in Fig. 39. If r is the distance from the axis of symmetry on which points A_1 and A_2 appear after compression, then, disregarding smalls of a higher order, one can assume that prior to the compression points A_1 and A_2 had coordinate z equal according to (66) to $z_1(r)$ and $-z_2(r)$. Then from Fig. 39 it follows that

$$a = u_{1z} + z_1(r) + z_2(r) - u_{2z} \qquad (67)$$

where u_{1z} and u_{2z} are projections of elastic displacements u_1 and u_2 on the z axis. Let us now turn to the calculation of these elastic displacement.

We will consider surfaces of compressibel bodies to be perfectly smooth and will designate by $p(r)$ the normal pressure appearing in the region of contact at distance r from the axis of symmetry. We will further approximately consider that unknown displacements u_{1z} and u_{2z} will be the same as if the pressure appearing in the region of contact acted on the upper and lower elastic half-spaces with the same elastic constants as those of the compressible bodies. In virtue of the axial symmetry, the region of contact will be the circle of a certain radius a unknown as yet (Fig. 40). Let us assume that $d\sigma$ is the element of area of this circle covering point A' located at distance r' from the axis of symmetry. Acting on this element of area will be the normal force $p(r')d\sigma$. As is known, the normal force P, which acts on the elastic half-space, causes at

distance R from the point of its application a normal displacement on the surface of the elastic medium equal[1] to $P\frac{\theta}{R}$, where $\theta = \frac{1-\mu^2}{\pi E}$, E — elastic modulus, and μ — Poisson's ratio.

Fig. 40.

Thus, the pressure acting on the element of area $d\sigma$ causes at point A, shown in Fig. 40, a normal elastic displacement du_z equal to

$$du_z = \theta\frac{p(r')\,d\sigma}{R},$$

where R — distance between points A and A'. In order to obtain the complete normal displacement u_z at point A, which is at the distance r from the axis of symmetry, it is necessary to integrate the elementary displacement du_z with respect to the whole area of contact. Let us find

$$u_z = \theta\iint\limits_{\Sigma}\frac{p(r')\,d\sigma}{R}, \tag{68}$$

if by Σ we designate the circle of the radius a, which represents the region of contact. Thus, the unknown displacements u_{1z} and u_{2z} will equal

$$u_{1z} = \theta_1\iint\limits_{\Sigma}\frac{p(r')\,d\sigma}{R}, \quad u_{2z} = -\theta_2\iint\limits_{\Sigma}\frac{p(r')\,d\sigma}{R}, \tag{69}$$

where

$$\theta_1 = \frac{1-\mu_1^2}{\pi E_1}, \quad \theta_2 = \frac{1-\mu_2^2}{\pi E_2}, \tag{70}$$

[1]See Timoshenko, S. P., Theory of elasticity, 1937, p. 364.

E_1 and E_2 – elastic moduli, and μ_1 and μ_2 – Poisson's ratio of compressible bodies.

Substituting (69) into (67), we will obtain the equation

$$(\vartheta_1+\vartheta_2)\iint_{\Sigma}\frac{p(r')\,ds}{R}=\alpha-z_1(r)-z_2(r),\quad 0<r<a, \tag{71}$$

or

$$\iint_{\Sigma}\frac{p(r')\,ds}{R}=f(r),\quad 0<r<a, \tag{72}$$

where

$$f(r)=\frac{\alpha-z_1(r)-z_2(r)}{\vartheta_1+\vartheta_2}. \tag{73}$$

Equation (72), obtained by us for the determination of pressure $p(r)$ in the region of contact, coincides with equation (1) of this chapter studied by us in § 1. As we showed in § 1, the solution of this equation is determined by formula (46):

$$p(r)=\frac{1}{2\pi}\left[\frac{F(a)}{\sqrt{a^2-r^2}}-\int_r^a\frac{F'(s)\,ds}{\sqrt{s^2-r^2}}\right],\quad 0<r<a, \tag{74}$$

where $F(r)$ is the function determined by formula (48) or formula (39). Substituting (73) into (48) and (39), we will find

$$F(r)=\frac{2}{\pi(\vartheta_1+\vartheta_2)}\left[\alpha-r\int_0^r\frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{r^2-\sigma^2}}\,d\sigma\right],\quad 0<r<a, \tag{75}$$

or

$$F(r)=\frac{2}{\pi(\vartheta_1+\vartheta_2)}\left\{\alpha-r\int_0^{\pi/2}[z_1'(r\sin\varphi)+z_2'(r\sin\varphi)]\,d\varphi\right\}, \tag{76}$$
$$0<r<a,$$

since $z_1(0)=z_2(0)=0$ (see Fig. 39). It is also possible to use formula (65) for the unknown pressure $p(r)$ in the region of contact.

The solution obtained by us contains two unknown constants: radius of the region of contact a and approach of bodies with compression α. Let us turn to their determination.

As can be seen from formula (74), if $F(a)\neq 0$, then pressure $p(r)\to\infty$, when $r\to a$. Thus, so that the expression found by us for pressure $p(r)$ remains limited in the whole region of contact, it is necessary that there is equality

$$F(a)=0. \tag{77}$$

If condition (77) is fulfilled, formula (74) takes the form

$$p(r)=-\frac{1}{2\pi}\int_r^a \frac{F'(s)\,ds}{\sqrt{s^2-r^2}}, \quad 0<r<a. \tag{78}$$

Pressure $p(r)$, determined by formula (78), on boundary of the region of contact, i.e., when $r = a$, becomes zero.

Substituting (75) and (76) into (77), we will obtain for the approach of bodies with compression α formulas

$$\alpha=a\int_0^a \frac{z_1'(\rho)+z_2'(\rho)}{\sqrt{a^2-\rho^2}}\,d\rho, \tag{79}$$

or

$$\alpha=a\int_0^{\pi/2}[z_1'(a\sin\varphi)+z_2'(a\sin\varphi)]\,d\varphi. \tag{80}$$

Let us designate further in terms of P the magnitude of the resultant of compressing forces. Pressure $p(r)$ appearing in the region of contact should balance force P, and consequently, integrating the pressure $p(r)$ with respect to the whole region of contact, we

should obtain force P. With the designations accepted by us we will obtain the condition

$$\iint_{\Sigma} p(r')\, d\sigma = P, \tag{81}$$

or, since the element of the area $d\sigma$ in polar coordinates r', ϕ' is equal to $r'dr'd\phi'$, we have

$$\int_0^{2\pi} d\varphi' \int_0^a p(r') r'\, dr' = P,$$

i.e.,

$$2\pi \int_0^a p(r') r'\, dr' = P. \tag{82}$$

Substituting (78) into (82), we will find

$$-\int_0^a dr' \int_{r'}^a \frac{r' F'(s)\, ds}{\sqrt{s^2 - r'^2}} = P. \tag{83}$$

If in the obtained multiple integral, variables of integration r' and s are examined as rectangular coordinates, the region of integration will be the triangle shaded in Fig. 41. As can be seen from the same figure, if one were to change the order of integration, relation (83) will take the form

$$-\int_0^a ds \int_0^s \frac{r' F'(s)\, dr'}{\sqrt{s^2 - r'^2}} = P, \tag{84}$$

or, since

$$\int_0^s \frac{r'\, dr'}{\sqrt{s^2 - r'^2}} = -\sqrt{s^2 - r'^2}\Big|_{r'=0}^{r'=s} = s,$$

we have

$$-\int_0^a F'(s)\, s\, ds = P. \tag{85}$$

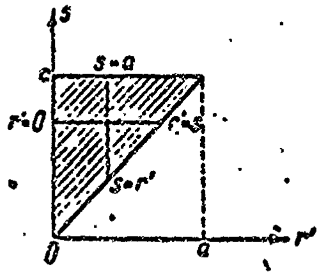

Fig. 41.

Fulfilling partial integration in (85), we obtain

$$\int_0^a F(s)\,ds - sF(s)\Big|_{s=0}^{s=a} = P,$$

or

$$\int_0^a F(s)\,ds = P \tag{86}$$

in virtue of condition (77).

Substituting (75) into (86), we find

$$a\alpha - \int_0^a ds \int_0^s s\frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{s^2-\sigma^2}}\,d\sigma = \frac{1}{2}\pi P(\theta_1+\theta_2). \tag{87}$$

Substituting (79) into (87), we obtain

$$a^2\int_0^a \frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{a^2-\sigma^2}}\,d\sigma - \int_0^a ds \int_0^s s\frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{s^2-\sigma^2}}\,d\sigma = \frac{1}{2}\pi P(\theta_1+\theta_2). \tag{88}$$

If in the multiple integral, which appears in formula (88), variables of integration s and σ are examined as rectangular coordinates, then the region of integration will be the triangle shaded in Fig. 42. If one were to change the order of integration, then, as can be seen from Fig. 42, we will have

$$\int_0^a ds \int_0^s s\frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{s^2-\sigma^2}}\,d\sigma = \int_0^a d\sigma \int_\sigma^a s\frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{s^2-\sigma^2}}\,ds, \tag{89}$$

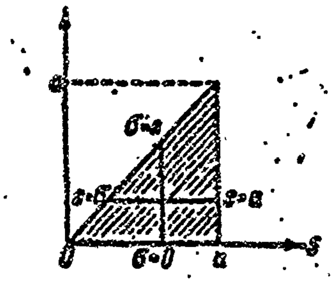

Fig. 42.

or, since

$$\int_\sigma^a \frac{s\,ds}{\sqrt{s^2-\sigma^2}} = \sqrt{s^2-\sigma^2}\,\Big|_{s=\sigma}^{s=a} = \sqrt{a^2-\sigma^2},$$

we obtain

$$\int_0^a ds \int_0^s s\,\frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{s^2-\sigma^2}}\,d\sigma = \int_0^a [z_1'(\sigma)+z_2'(\sigma)]\sqrt{a^2-\sigma^2}\,d\sigma. \qquad (90)$$

Substituting (90) into (88), we find

$$\int_0^a [z_1'(\sigma)+z_2'(\sigma)]\left(\frac{a^2}{\sqrt{a^2-\sigma^2}}-\sqrt{a^2-\sigma^2}\right)d\sigma = \frac{1}{2}\pi P(\vartheta_1+\vartheta_2),$$

or finally

$$\int_0^a \frac{z_1'(\sigma)+z_2'(\sigma)}{\sqrt{a^2-\sigma^2}}\,\sigma^2\,d\sigma = \frac{1}{2}\pi P(\vartheta_1+\vartheta_2). \qquad (91)$$

Assuming in (91) $\sigma = a \sin \phi$, it is possible to give to relation (91) the form

$$a^2\int_0^{\pi/2} [z_1'(a\sin\varphi)+z_2'(a\sin\varphi)]\sin^2\varphi\,d\varphi = \frac{1}{2}\pi P(\vartheta_1+\vartheta_2). \qquad (92)$$

Thus, the radius of the region of contact a is determined by equation (91) or equation (92) equivalent to it. After constant a is found, the approach of bodies with compression α can be calculated by formula (79) or formula (80). The distribution of pressure p in

region of contact will be determined by formula (78), where $F(r)$ is the function determined by formula (75) or formula (76) equivalent to it.

The formulas enumerated by us give the general solution of the problem about the compression of two elastic bodies limited by surfaces of rotation. Let us now turn to analysis of different particular cases of axisymmetrical contact problem of theory of elasticity.

First let us assume that the initial contact of compressible elastic bodies is carried out at one point, and for surfaces of both bodies this point is regular. In this case functions $z_1(r)$ and $z_2(r)$, which determine the configuration of the compressible bodies, can be expanded in Taylor series in neighborhood of point $r = 0$:

$$\left.\begin{aligned} z_1(r) &= z_1(0) + z_1'(0)r + \frac{1}{2!}z_1''(0)r^2 + \frac{1}{3!}z_1'''(0)r^3 + \ldots, \\ z_2(r) &= z_2(0) + z_2'(0)r + \frac{1}{2!}z_2''(0)r^2 + \frac{1}{3!}z_2'''(0)r^3 + \ldots \end{aligned}\right\} \quad (93)$$

Since we disposed the origin of cylindrical coordinates r, ϕ, z at the point of contact of the compressible bodies and axis z is perpendicular to the plane tangent to both surfaces of compressible bodies at the point of their contact, we will have

$$z_1(0) = z_2(0) = 0, \quad z_1'(0) = z_2'(0) = 0. \quad (94)$$

Thus, according to (93) and (94) for the sum of functions $z_1(r) + z_2(r)$ we will have the expansion

$$z_1(r) + z_2(r) = \frac{z_1''(0) + z_2''(0)}{2!}r^2 + \frac{z_1'''(0) + z_2'''(0)}{3!}r^3 + \ldots \quad (95)$$

Let us first examine the case when the sum of the second derivatives

$$z_1''(0) + z_2''(0) \neq 0. \quad (96)$$

In view of the smallness of elastic displacements and hence resulting smallness of the region of contact, it is possible to take for the sum of functions $z_1(r) = z_2(r)$ in the region of contact the approximat expression

$$z_1(r)+z_2(r)=\frac{1}{2}[z_1''(0)+z_2''(0)]r^2, \quad 0<r<a, \tag{97}$$

dropping in the expansion of (95) terms of the highest orders of smallness. Introducing the designation

$$A=\frac{1}{2}[z_1''(0)+z_2''(0)], \tag{98}$$

we will have in the examined case

$$z_1(r)+z_2(r)=Ar^2, \quad 0<r<a. \tag{99}$$

Substituting (99) into (92), we will obtain the relation for the determination of the radius of the region of contact a

$$2Aa^3\int_0^{\pi/2}\sin^3\varphi\,d\varphi=\frac{1}{2}\pi P(\vartheta_1+\vartheta_2). \tag{100}$$

Since

$$\int_0^{\pi/2}\sin^3\varphi\,d\varphi=-\int_0^{\pi/2}(1-\cos^2\varphi)\,d(\cos\varphi)=-\cos\varphi+\frac{1}{3}\cos^3\varphi\Big|_{\varphi=0}^{\varphi=\frac{\pi}{2}}=\frac{2}{3},$$

from (100) we find

$$a=\frac{1}{2}\sqrt[3]{\frac{3\pi P(\vartheta_1+\vartheta_2)}{A}}. \tag{101}$$

Substituting (99) into (80), we obtain

$$\alpha=2Aa^2\int_0^{\pi/2}\sin\varphi\,d\varphi=2Aa^2. \tag{102}$$

Substituting (101) into (102), we find

$$\alpha = \frac{1}{2}\sqrt[3]{9\pi^2 A P^2 (\vartheta_1+\vartheta_2)^2}. \tag{103}$$

Substituting (99) into (76), we obtain

$$F(r) = \frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(\alpha - 2Ar^2\int_0^{\pi/2}\sin\varphi\,d\varphi\right) = \frac{2(\alpha-2Ar^2)}{\pi(\vartheta_1+\vartheta_2)}. \tag{104}$$

Substituting (104) into (78), we find

$$p(r) = \frac{4A}{\pi^2(\vartheta_1+\vartheta_2)}\int_r^a \frac{s\,ds}{\sqrt{s^2-r^2}} = \frac{4A}{\pi^2(\vartheta_1+\vartheta_2)}\sqrt{s^2-r^2}\Big|_{s=r}^{s=a} = \frac{4A\sqrt{a^2-r^2}}{\pi^2(\vartheta_1+\vartheta_2)}, \quad 0<r<a. \tag{105}$$

But according to (101)

$$\frac{A}{\vartheta_1+\vartheta_2} = \frac{3\pi P}{8a^3}. \tag{106}$$

Substituting (106) into (105), we find

$$p(r) = \frac{3}{2}\sqrt{1-\frac{r^2}{a^2}}\,\frac{P}{\pi a^2}, \quad 0<r<a. \tag{107}$$

As we see, pressure p in the region of contact changes in the axisymmetrical contact problem, depending upon the distance to the initial point of contact, according to the same law as that in the two-dimensional contact problem of the theory of elasticity.

Let us consider now the special case of the contact problem when the sum of the second derivatives $z_1''(r)+z_2''(r)$ becomes zero when $r = 0$:

$$z_1''(0)+z_2''(0)=0. \tag{108}$$

For generality we will assume that first term of expansion (95) not turning into zero contains factor r in power $2n$ (coefficients at all odd powers of r in this expansion must be equal to zero due to the fact that functions $z_1(r)$ and $z_2(r)$ are even, if one were to examine them not only for positive but also for negative values of the argument r). Then, disregarding in expansion (95) terms of the highest order of smallness, in region of contact we will have

$$z_1(r)+z_2(r)=\frac{z_1^{(2n)}(0)+z_2^{(2n)}(0)}{(2n)!}r^{2n},$$

or

$$z_1(r)+z_2(r)=Ar^{2n}, \tag{109}$$

if one were to introduce designation

$$A=\frac{1}{(2n)!}[z_1^{(2n)}(0)+z_2^{(2n)}(0)]. \tag{110}$$

Substituting (109) into (92), we find:

$$2nAa^{2n+1}\int_0^{\pi/2}\sin^{2n+1}\varphi\,d\varphi=\frac{1}{2}\pi P(\vartheta_1+\vartheta_2). \tag{111}$$

Let us introduce the designation

$$c_n=\int_0^{\pi/2}\sin^{2n+1}\varphi\,d\varphi. \tag{112}$$

By means of integration by parts we find

$$\begin{aligned}c_n&=\int_0^{\pi/2}\sin^{2n+1}\varphi\,d\varphi=-\int_0^{\pi/2}\sin^{2n}\varphi\,d(\cos\varphi)=\\&=2n\int_0^{\pi/2}\sin^{2n-1}\varphi\cos^2\varphi\,d\varphi-\sin^{2n}\varphi\cos\varphi\Big|_0^{\pi/2}=\\&=2n\int_0^{\pi/2}\sin^{2n-1}\varphi(1-\sin^2\varphi)\,d\varphi=\\&=2n\int_0^{\pi/2}\sin^{2n-1}\varphi\,d\varphi-2n\int_0^{\pi/2}\sin^{2n+1}\varphi\,d\varphi=2nc_{n-1}-2nc_n,\end{aligned}$$

whence

$$c_n = \frac{2n}{2n+1} c_{n-1}. \tag{113}$$

Assuming in (113) $n - 1$ instead of n, we obtain

$$c_{n-1} = \frac{2n-2}{2n-1} c_{n-2}. \tag{114}$$

Substituting (114) into (113), we find

$$c_n = \frac{2n(2n-2)}{(2n+1)(2n-1)} c_{n-2}. \tag{115}$$

Expressing in (115) c_{n-2} in terms of c_{n-3} on the basis of the general relation (113), we obtain

$$c_n = \frac{2n(2n-2)(2n-4)}{(2n+1)(2n-1)(2n-3)} c_{n-3}.$$

Continuing the indicated process, we will arrive at formula

$$c_n = \frac{2n(2n-2)(2n-4)\ldots 4\cdot 2}{(2n+1)(2n-1)(2n-3)\ldots 5\cdot 3} c_0. \tag{116}$$

Assuming in (112) $n = 0$, we find

$$c_0 = \int_0^{\pi/2} \sin\varphi\, d\varphi = 1. \tag{117}$$

Substituting (117) into (116), we obtain

$$c_n = \frac{2\cdot 4\cdot 6\ldots (2n-2)\, 2n}{3\cdot 5\cdot 7\ldots (2n-1)(2n+1)}. \tag{118}$$

Substituting (118) into (111), we find

$$2nAa^{2n+1}\frac{2\cdot 4\cdot 6\ldots (2n-2)\, 2n}{3\cdot 5\cdot 7\ldots (2n-1)(2n+1)} = \frac{1}{2}\pi P(\vartheta_1 + \vartheta_2),$$

whence

$$a = \sqrt[2n+1]{\frac{\pi}{4n}\frac{3\cdot5\cdot7\ldots(2n-1)(2n+1)}{2\cdot4\cdot6\ldots(2n-2)2n}\frac{P(\vartheta_1+\vartheta_2)}{A}}. \tag{119}$$

Substituting (109) into (80), we find

$$\alpha = 2nAa^{2n}\int_0^{\pi/2}\sin^{2n-1}\varphi\,d\varphi = 2nAa^{2n}c_{n-1}, \tag{120}$$

in accordance with designation (112). Substituting (118) into (120), we obtain

$$\alpha = \frac{2\cdot4\cdot6\ldots(2n-4)(2n-2)2n}{3\cdot5\cdot7\ldots(2n-3)(2n-1)}Aa^{2n}; \tag{121}$$

Substituting (109) into (76), we will find

$$F(r) = \frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(\alpha - 2nAr^{2n}\int_0^{\pi/2}\sin^{2n-1}\varphi\,d\varphi\right) = \frac{2(\alpha-2nAr^{2n}c_{n-1})}{\pi(\vartheta_1+\vartheta_2)}. \tag{122}$$

Substituting (118) into (122), we obtain

$$F(r) = \frac{2}{\pi(\vartheta_1+\vartheta_2)}\left[\alpha - \frac{2\cdot4\cdot6\ldots(2n-4)(2n-2)2n}{3\cdot5\cdot7\ldots(2n-3)(2n-1)}Ar^{2n}\right]; \tag{123}$$

Substituting (123) into (78), we find

$$p(r) = \frac{2\cdot4\cdot6\ldots(2n-4)(2n-2)2n}{3\cdot5\cdot7\ldots(2n-3)(2n-1)}\frac{2nA}{\pi^2(\vartheta_1+\vartheta_2)}\int_r^a\frac{s^{2n-1}ds}{\sqrt{s^2-r^2}}. \tag{124}$$

Assuming in (124) $s = a\sigma$, we obtain

$$p(r) = \frac{2\cdot4\cdot6\ldots(2n-4)(2n-2)2n}{3\cdot5\cdot7\ldots(2n-3)(2n-1)}\frac{2nAa^{2n-1}}{\pi^2(\vartheta_1+\vartheta_2)}\int_{r/a}^1\frac{\sigma^{2n-1}d\sigma}{\sqrt{\sigma^2-\left(\frac{r}{a}\right)^2}}. \tag{125}$$

Introducing designation

$$p_n(\rho) = \int_\rho^1 \frac{\sigma^{2n-1}\,d\sigma}{\sqrt{\sigma^2-\rho^2}}, \tag{126}$$

we will be able to give to formula (125) the form

$$p(r) = \frac{2\cdot 4\cdot 6\ldots(2n-4)(2n-2)\,2n}{3\cdot 5\cdot 7\ldots(2n-3)(2n-1)}\,\frac{2nAa^{2n-1}}{\pi^2(\vartheta_1+\vartheta_2)}\,p_n\left(\frac{r}{a}\right). \tag{127}$$

By means of integration by parts we find

$$\begin{aligned}
p_n(\rho) &= \int_\rho^1 \frac{\sigma^{2n-1}\,d\sigma}{\sqrt{\sigma^2-\rho^2}} = \int_\rho^1 \sigma^{2n-2}\,d\left(\sqrt{\sigma^2-\rho^2}\right) = \\
&= -(2n-2)\int_\rho^1 \sigma^{2n-3}\sqrt{\sigma^2-\rho^2}\,d\sigma + \sigma^{2n-2}\sqrt{\sigma^2-\rho^2}\Big|_{\sigma=\rho}^{\sigma=1} = \\
&= -(2n-2)\int_\rho^1 \frac{\sigma^{2n-1}-\rho^2\sigma^{2n-3}}{\sqrt{\sigma^2-\rho^2}}\,d\sigma + \sqrt{1-\rho^2} = \\
&= -(2n-2)\int_\rho^1 \frac{\sigma^{2n-1}\,d\sigma}{\sqrt{\sigma^2-\rho^2}} + (2n-2)\rho^2\int_\rho^1 \frac{\sigma^{2n-3}\,d\sigma}{\sqrt{\sigma^2-\rho^2}} + \sqrt{1-\rho^2} = \\
&= -(2n-2)\,p_n(\rho) + (2n-2)\rho^2 p_{n-1}(\rho) + \sqrt{1-\rho^2},
\end{aligned}$$

whence

$$p_n(\rho) = \frac{2n-2}{2n-1}\rho^2 p_{n-1}(\rho) + \frac{\sqrt{1-\rho^2}}{2n-1}. \tag{128}$$

Assuming in (128) $n - 1$ instead of n, we find

$$p_{n-1}(\rho) = \frac{2n-4}{2n-3}\rho^2 p_{n-2}(\rho) + \frac{\sqrt{1-\rho^2}}{2n-3}; \tag{129}$$

Substituting (129) into (128), we have

$$p_n(\rho) = \sqrt{1-\rho^2}\left[\frac{1}{2n-1} + \frac{2n-2}{(2n-1)(2n-3)}\rho^2\right] + \frac{(2n-2)(2n-4)}{(2n-1)(2n-3)}\rho^4 p_{n-2}(\rho). \tag{130}$$

Assuming in (128) $n - 2$ instead of n, we obtain

$$p_{n-2}(\rho)=\frac{2n-6}{2n-5}\rho^2 p_{n-3}(\rho)+\frac{\sqrt{1-\rho^2}}{2n-5}. \tag{131}$$

Substituting (131) into (130), we find

$$p_n(\rho)=\sqrt{1-\rho^2}\left[\frac{1}{2n-1}+\frac{2n-2}{(2n-1)(2n-3)}\rho^2+\right.$$
$$\left.+\frac{(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}\rho^4\right]+\frac{(2n-2)(2n-4)(2n-6)}{(2n-1)(2n-3)(2n-5)}\rho^6 p_{n-3}(\rho). \tag{132}$$

Continuing the indicated process, we arrive at the formula

$$p_n(\rho)=\sqrt{1-\rho^2}\left[\frac{1}{2n-1}+\frac{2n-2}{(2n-1)(2n-3)}\rho^2+\frac{(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}\rho^4+\right.$$
$$\left.+\dots+\frac{(2n-2)(2n-4)\dots 6\cdot 4}{(2n-1)(2n-3)\dots 7\cdot 5\cdot 3}\rho^{2n-4}\right]+$$
$$+\frac{(2n-2)(2n-4)\dots 6\cdot 4\cdot 2}{(2n-1)(2n-3)\dots 7\cdot 5\cdot 3}\rho^{2n-2}p_1(\rho). \tag{133}$$

Considering in (126) $n = 1$, we find

$$p_1(\rho)=\int_\rho^1\frac{\sigma\,d\sigma}{\sqrt{\sigma^2-\rho^2}}=\sqrt{\sigma^2-\rho^2}\,\Big|_{\sigma=\rho}^{\sigma=1}=\sqrt{1-\rho^2}. \tag{134}$$

Substituting (134) into (133), we obtain

$$p_n(\rho)=\sqrt{1-\rho^2}\left[\frac{1}{2n-1}+\frac{2n-2}{(2n-1)(2n-3)}\rho^2+\frac{(2n-2)(2n-4)}{(2n-1)(2n-3)(2n-5)}\rho^4+\right.$$
$$\left.+\dots+\frac{(2n-2)(2n-4)\dots 6\cdot 4}{(2n-1)(2n-3)\dots 7\cdot 5\cdot 3}\rho^{2n-4}+\frac{(2n-2)(2n-4)\dots 4\cdot 2}{(2n-1)(2n-3)\dots 5\cdot 3\cdot 1}\rho^{2n-2}\right],$$

or

$$p_n(\rho)=\frac{(2n-2)(2n-4)\dots 4\cdot 2}{(2n-1)(2n-3)\dots 5\cdot 3}\left[\rho^{2n-2}+\frac{1}{2}\rho^{2n-4}+\frac{3}{2\cdot 4}\rho^{2n-6}+\dots+\right.$$
$$\left.+\frac{3\cdot 5\cdot 7\dots(2n-7)(2n-5)}{2\cdot 4\cdot 6\dots(2n-6)(2n-4)}\rho^2+\frac{3\cdot 5\cdot 7\dots(2n-5)(2n-3)}{2\cdot 4\cdot 6\dots(2n-4)(2n-2)}\right]\sqrt{1-\rho^2}. \tag{135}$$

Substituting (135) into (127), we find

$$p(r)=\left[\frac{2\cdot 4\cdot 6\ldots(2n-2)\,2n}{3\cdot 5\cdot 7\ldots(2n-3)(2n-1)}\right]^2 \frac{Aa^{2n-1}}{\pi^2(\vartheta_1+\vartheta_2)}\left[\left(\frac{r}{a}\right)^{2n-2}+\right.$$
$$+\frac{1}{2}\left(\frac{r}{a}\right)^{2n-4}+\frac{3}{2\cdot 4}\left(\frac{r}{a}\right)^{2n-6}+\cdots+\frac{3\cdot 5\cdot 7\ldots(2n-7)(2n-5)}{2\cdot 4\cdot 6\ldots(2n-6)(2n-4)}\left(\frac{r}{a}\right)^2+$$
$$\left.+\frac{3\cdot 5\cdot 7\ldots(2n-5)(2n-3)}{2\cdot 4\cdot 6\ldots(2n-4)(2n-2)}\right]\sqrt{1-\left(\frac{r}{a}\right)^2}, \qquad 0<r<a. \tag{136}$$

From (119) it follows that

$$\frac{Aa^{2n-1}}{\pi^2(\vartheta_1+\vartheta_2)}=\frac{1}{4n}\,\frac{3\cdot 5\cdot 7\ldots(2n-1)(2n+1)}{2\cdot 4\cdot 6\ldots(2n-2)\,2n}\,\frac{P}{\pi a^2}. \tag{137}$$

Substituting (137) into (136), we obtain

$$p(r)=\frac{2\cdot 4\cdot 6\ldots(2n-4)(2n-2)}{3\cdot 5\cdot 7\ldots(2n-3)(2n-1)}\,\frac{2n+1}{2}\left[\left(\frac{r}{a}\right)^{2n-2}+\frac{1}{2}\left(\frac{r}{a}\right)^{2n-4}+\right.$$
$$+\frac{3}{2\cdot 4}\left(\frac{r}{a}\right)^{2n-6}+\cdots+\frac{3\cdot 5\cdot 7\ldots(2n-7)(2n-5)}{2\cdot 4\cdot 6\ldots(2n-6)(2n-4)}\left(\frac{r}{a}\right)^2+$$
$$\left.+\frac{3\cdot 5\cdot 7\ldots(2n-5)(2n-3)}{2\cdot 4\cdot 6\ldots(2n-4)(2n-2)}\right]\sqrt{1-\left(\frac{r}{a}\right)^2}\,\frac{P}{\pi a^2}, \qquad 0<r<a. \tag{138}$$

Thus, formula (119) determines the radius of the region of contact a; after constant a is found formula (121) will enable us to calculate the approach of bodies with compression α, and formula (138) determines pressure p in the region of contact. Pressure p changes depending upon the distance to the initial point of contact according to the same law as that in the corresponding two-dimensional contact problem (see formula (46) and Fig. 9 of Chapter II).

Until now we assumed that the point of the initial contact of the compressible bodies is the regular point of surfaces of both bodies. Let us now turn to the consideration of the case when for the surface of one of the compressible bodies or for surfaces of both bodies subjected to compression the point of initial contact is a singular point.

Let us examine first the case when the point of initial contact of compressible bodies is the corner point of the axial section of the surface of one of the bodies subjected to compression (Fig. 43). If the tangent to the generatrix of this surface in the corner point forms with axis z angle γ, then, as can be seen from Fig. 43, disregarding smalls of higher orders, in the neighborhood of the origin of the coordinates we will have

$$z_1(r)+z_2(r)=r\,\mathrm{ctg}\,\gamma. \tag{139}$$

Substituting (139) in (92), we find

$$a^2\,\mathrm{ctg}\,\gamma\int\limits_0^{\pi/2}\sin^2\varphi\,d\varphi=\frac{1}{2}\pi P(\vartheta_1+\vartheta_2),$$

or, since

$$\int\limits_0^{\pi/2}\sin^2\varphi\,d\varphi=\frac{1}{2}\int\limits_0^{\pi/2}(1-\cos 2\varphi)\,d\varphi=\frac{1}{2}\left(\varphi-\frac{1}{2}\sin 2\varphi\right)_0^{\pi/2}=\frac{\pi}{4},$$

$$\frac{\pi}{4}a^2\,\mathrm{ctg}\,\gamma=\frac{1}{2}\pi P(\vartheta_1+\vartheta_2),$$

whence

$$a=\sqrt{2P(\vartheta_1+\vartheta_2)\,\mathrm{tg}\,\gamma}. \tag{140}$$

Substituting (139) into (80), we find

$$\alpha=\frac{\pi}{2}a\,\mathrm{ctg}\,\gamma. \tag{141}$$

Substituting (139) into (76), we obtain

$$F(r)=\frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(\alpha-\frac{\pi}{2}r\,\mathrm{ctg}\,\gamma\right). \tag{142}$$

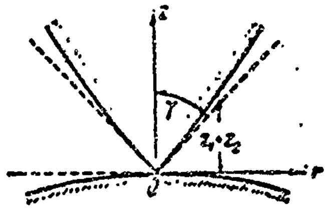

Fig. 43.

Substituting (142) into (78), we find

$$p(r)=\frac{\operatorname{ctg}\gamma}{2\pi(\vartheta_1+\vartheta_2)}\int_r^a\frac{ds}{\sqrt{s^2-r^2}}=\frac{\operatorname{ctg}\gamma}{2\pi(\vartheta_1+\vartheta_2)}\ln\left(s-\sqrt{s^2-r^2}\right)\Big|_{s=a}^{s=r}=$$
$$=-\frac{\operatorname{ctg}\gamma}{2\pi(\vartheta_1+\vartheta_2)}\ln\left(\frac{a}{r}-\sqrt{\frac{a^2}{r^2}-1}\right). \qquad (143)$$

From (140) it follows that

$$\frac{\operatorname{ctg}\gamma}{\vartheta_1+\vartheta_2}=\frac{2P}{a^2}. \qquad (144)$$

Substituting (144) into (143), we find

$$p(r)=-\frac{P}{\pi a^2}\ln\left(\frac{a}{r}-\sqrt{\frac{a^2}{r^2}-1}\right),\quad 0<r<a. \qquad (145)$$

In the examined case again we obtain the same dependence of pressure p on distance up to the initial point of contact, as that in the corresponding two-dimensional contact problem (see formula (107) and Fig. 12a of Chapter II). Pressure $p(r)$ increases without limit when distance r up to the initial point of contact approaches zero.

In conclusion of this section let us examine the case when axial sections of surfaces of compressible bodies have at the point of initial contact a continuously revolving tangent, but the curvature of one or both indicated sections at this point is infinitely great. We restrict ourselves to an examination of the example in which the initial distance between the points touching with compression $z_1 + z_2$ can be represented in the neighborhood of the origin of coordinates by the relation

$$z_1(r)+z_2(r) = Ar^{3/2}, \quad 0<r<a. \tag{146}$$

Substituting (146) into (91), we find

$$\int_0^a \frac{\sigma^2\sqrt{\sigma}\,d\sigma}{\sqrt{a^2-\sigma^2}} = \frac{\pi P(\vartheta_1+\vartheta_2)}{3A}. \tag{147}$$

Assuming in (147) $\sigma = at$, we obtain

$$a^{5/2}\int_0^1 \frac{t^2\sqrt{t}\,dt}{\sqrt{1-t^2}} = \frac{\pi P(\vartheta_1+\vartheta_2)}{3A}. \tag{148}$$

Substituting (146) into (79), we find

$$\alpha = \frac{3}{2}Aa\int_0^a \frac{\sqrt{\sigma}\,d\sigma}{\sqrt{a^2-\sigma^2}},$$

or, assuming $\sigma = at$,

$$\alpha = \frac{3}{2}Aa^{3/2}\int_0^1 \frac{\sqrt{t}\,dt}{\sqrt{1-t^2}}. \tag{149}$$

Definite integrals entering into formulas (148) and (149) are elliptic, and after reduction to canonical form they can be calculated with the help of tables for elliptic integrals. Their values are such[1]

$$\int_0^1 \frac{\sqrt{t}\,dt}{\sqrt{1-t^2}} = J_1 = 1{,}1981, \quad \int_0^1 \frac{t^2\sqrt{t}\,dt}{\sqrt{1-t^2}} = J_2 = 0{,}7189. \tag{150}$$

Substituting (150) into (148) and (149), we will obtain the final formulas

[1]See Appendix 1, formulas (1) and (14).

$$a = \left[\frac{\pi P(\vartheta_1+\vartheta_2)}{3J_3 A}\right]^{2/3}, \tag{151}$$

$$\alpha = \frac{3}{2} J_1 A a^{3/2}. \tag{152}$$

Substituting (146) into (75), we find

$$F(r) = \frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(\alpha - \frac{3}{2} A r \int_0^r \frac{\sqrt{\sigma}\,d\sigma}{\sqrt{r^2-\sigma^2}}\right). \tag{153}$$

Considering in (153) $\sigma = rt$, we obtain

$$F(r) = \frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(\alpha - \frac{3}{2} A r^{3/2} \int_0^1 \frac{\sqrt{t}\,dt}{\sqrt{1-t^2}}\right),$$

or, according to (150),

$$F(r) = \frac{2}{\pi(\vartheta_1+\vartheta_2)}\left(\alpha - \frac{3}{2} J_1 A r^{3/2}\right). \tag{154}$$

Substituting (154) into (78), we find

$$p(r) = \frac{9J_1 A}{4\pi^2(\vartheta_1+\vartheta_2)} \int_r^a \frac{\sqrt{s}\,ds}{\sqrt{s^2-r^2}}, \qquad 0<r<a. \tag{155}$$

Assuming in (155) $s = r/t$, we obtain

$$p(r) = \frac{9J_1 A\sqrt{r}}{4\pi^2(\vartheta_1+\vartheta_2)} \int_{r/a}^1 \frac{dt}{t\sqrt{t(1-t^2)}}, \qquad 0<r<a. \tag{156}$$

Introducing designation

$$f(\rho) = \sqrt{\rho} \int_\rho^1 \frac{dt}{t\sqrt{t(1-t^2)}}, \qquad 0<\rho<1, \tag{157}$$

we will be able to give to formula (156) the form

$$p(r) = \frac{9J_1 A\sqrt{a}}{4\pi^2(\vartheta_1+\vartheta_2)} f\left(\frac{r}{a}\right), \qquad 0<r<a. \tag{158}$$

From relation (151) it follows that

$$\frac{A}{\vartheta_1+\vartheta_2}=\frac{\pi P}{8J_2a^2\sqrt{a}}. \tag{159}$$

Substituting (159) into (158) and assuming according to (150) $\frac{J_2}{J_1}=0.\dot{6}$, we find

$$p(r)=1.25\frac{P}{\pi a^2}f\left(\frac{r}{a}\right), \quad 0<r<a. \tag{160}$$

The definite integral, which enters into formula (157) and determines function $f(\rho)$, is elliptic and after reduction to canonical form can be calculated as a function of variable limit ρ with the help of tables for elliptic integrals[1]. As calculations show, dependences of pressure p on distance r to the initial point of contact, determined by formula (160), is the same as that for the corresponding two-dimensional contact problem examined by us in Chapter II (see Fig. 11). Pressure p remains limited in the whole region of contact; however, derivative dp/dr increases without limit according to the absolute value when r approaches zero.

§ 3. Pressure of a Round Cylindrical Stamp on an Elastic Half-Space

The problem about the pressure of a rigid stamp on an elastic half-space, which is the subject of this section, differs from the contact problems examined in the preceding section by the fact that in this problem the region of contact is predetermined by the shape of the stamp. If one were to designate a as the radius of the base of the stamp (Fig. 44), the region of contact of the stamp with the elastic medium will always be a circle of radius a, independently of what force is pressing the stamp to the elastic half-space.

The initial distance between points of the compressible bodies, which touch with compression, which we designated in § 2 by $z_1 + z_2$,

[1]See Appendix 1, p. 7.

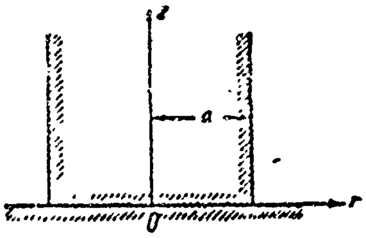

Fig. 44.

will in the examined problem be equal to zero in the whole region of contact:

$$z_1(r)+z_2(r)=0, \qquad 0<r<a. \tag{161}$$

Substituting (161) into (75), we obtain

$$F(r)=\frac{2\alpha}{\pi(\vartheta_1+\vartheta_2)}, \qquad 0<r<a. \tag{162}$$

Substituting (162) into (74), we find

$$p(r)=\frac{\alpha}{\pi^2(\vartheta_1+\vartheta_2)\sqrt{a^2-r^2}}, \qquad 0<r<a. \tag{163}$$

Substituting (163) into condition (82), we obtain

$$\frac{2\alpha}{\pi(\vartheta_1+\vartheta_2)}\int_0^a\frac{r'\,dr'}{\sqrt{a^2-r'^2}}=P,$$

or, since

$$\int_0^a\frac{r'\,dr'}{\sqrt{a^2-r'^2}}=-\sqrt{a^2-r'^2}\,\Big|_0^a=a,$$

$$\frac{2a\alpha}{\pi(\vartheta_1+\vartheta_2)}=P.$$

The obtained relation determines the approximation

$$\alpha=\frac{\pi P(\vartheta_1+\vartheta_2)}{2a}. \tag{164}$$

Substituting (164) into (163), we find

$$p(r) = 0{,}5\,\frac{P}{\pi a^2}\,\frac{1}{\sqrt{1-\frac{r^2}{a^2}}},$$
$$0 < r < a. \qquad (165)$$

We obtained the same dependence of pressure p on distance r up to the axis of the stamp, which in the corresponding two-dimensional contact problem (see formula (116) and Fig. 13 of Chapter II). On the boundary of the base of the stamp (when $r = a$), according to the formula (165), an infinitely large pressure p appears.

In order to obtain a picture of the distribution of pressure under the stamp close to the real one, just as in Chapter II, we will assume that the edge of the stamp has limited curvature even though it is large (Fig. 45).

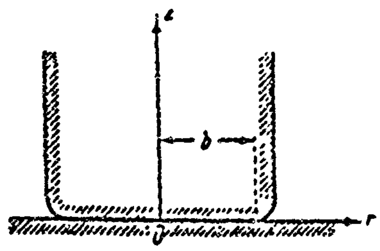

Fig. 45.

If by b we designate the radius of the circle, in the region of which initial contact of the stamp is carried out with the elastic half-space, then with compression the radius of the region of contact a will take a value somewhat exceeding its initial value b.

For the initial distance $z_1 + z_2$ between points touching upon compression will have the expression

$$\left.\begin{array}{l} z_1(r) + z_2(r) = 0, \quad 0 < r < b, \\ z_1(r) + z_2(r) = A(r-b)^2, \quad b < r < a, \end{array}\right\} \qquad (166)$$

if smalls of higher orders are disregarded.

Substituting (166) into (91), we find

$$2A\int_b^a \frac{(\sigma-b)\,\sigma^2\,d\sigma}{\sqrt{a^2-\sigma^2}} = \frac{1}{3}\,\pi P(\vartheta_1+\vartheta_2). \tag{167}$$

Assuming $\sigma = a\cos\phi$, we have

$$\int_b^a \frac{(\sigma-b)\,\sigma^2\,d\sigma}{\sqrt{a^2-\sigma^2}} = a^2\int_0^{\varphi_0}(a\cos\varphi-b)\cos^2\varphi\,d\varphi =$$

$$= a^2\left[a\int_0^{\varphi_0}(1-\sin^2\varphi)\,d(\sin\varphi)-\frac{b}{2}\int_0^{\varphi_0}(1+\cos 2\varphi)\,d\varphi\right] =$$

$$= a^2\left[a\left(\sin\varphi_0-\frac{1}{3}\sin^3\varphi_0\right)-\frac{b}{2}\left(\varphi_0+\frac{1}{2}\sin 2\varphi_0\right)\right], \tag{168}$$

where

$$\varphi_0 = \arccos\frac{b}{a}. \tag{169}$$

According to (169)

$$a = \frac{b}{\cos\varphi_0}. \tag{170}$$

Substituting (170) into (168), we find

$$\int_b^a \frac{(\sigma-b)\,\sigma^2\,d\sigma}{\sqrt{a^2-\sigma^2}} =$$

$$= b^3\sec^3\varphi_0\left[\sin\varphi_0-\frac{1}{3}\sin^3\varphi_0-\frac{1}{2}\cos\varphi_0\,(\varphi_0+\sin\varphi_0\cos\varphi_0)\right] =$$

$$= \frac{b^3}{6}\sec^3\varphi_0\,[\sin\varphi_0\,(6-2\sin^2\varphi_0-3\cos^2\varphi_0)-3\varphi_0\cos\varphi_0] =$$

$$= \frac{b^3}{6}\sec^3\varphi_0\,(3\sin\varphi_0+\sin^3\varphi_0-3\varphi_0\cos\varphi_0). \tag{171}$$

Substituting (171) into (167), we obtain the equation

$$\sec^3\varphi_0\,(3\sin\varphi_0+\sin^3\varphi_0-3\varphi_0\cos\varphi_0) = \frac{3\pi P(\vartheta_1+\vartheta_2)}{2Ab^3}, \tag{172}$$

which determines angle ϕ_0. Having found ϕ_0, we find by formula (170) the radius of the region of contact a.

Substituting (166) into (79), we find

$$\alpha = 2Aa \int_b^a \frac{\sigma - b}{\sqrt{a^2 - \sigma^2}}\, d\sigma,$$

or, assuming $\sigma = a \cos \phi$,

$$\alpha = 2Aa \int_0^{\varphi_0} (a \cos \varphi - b)\, d\varphi = 2Aa (a \sin \varphi_0 - b\varphi_0). \qquad (173)$$

Substituting (170) into (173), we obtain

$$\alpha = 2Ab^2 \sec \varphi_0 (\mathrm{tg}\, \varphi_0 - \varphi_0). \qquad (174)$$

Having found angle ϕ_0 from equation (172), we find by formula (174), the approach α.

Substituting (166) into (75), we find

$$F(r) = \frac{2\alpha}{\pi(\theta_1 + \theta_2)} \text{ when } 0 < r < b,$$
$$F(r) = \frac{2}{\pi(\theta_1 + \theta_2)} \left(\alpha - 2Ar \int_b^r \frac{\sigma - b}{\sqrt{r^2 - \sigma^2}}\, d\sigma \right) \text{ when } b < r < a,$$

or, since

$$\int_b^r \frac{\sigma - b}{\sqrt{r^2 - \sigma^2}}\, d\sigma = \left(-\sqrt{r^2 - \sigma^2} + b \arccos \frac{\sigma}{r} \right)_{\sigma = b}^{\sigma = r} = \sqrt{r^2 - b^2} - b \arccos \frac{b}{r};$$

$$\left.\begin{aligned} F(r) &= \frac{2\alpha}{\pi(\theta_1 + \theta_2)} \text{ when } 0 < r < b, \\ F(r) &= \frac{2}{\pi(\theta_1 + \theta_2)} \left(\alpha - 2Ar\sqrt{r^2 - b^2} + 2Ar\, b \arccos \frac{b}{r} \right) \\ &\qquad \text{when } b < r < a. \end{aligned}\right\} \qquad (175)$$

Fulfilling in (175) differentiations with respect to r, we find

$$F'(r)=0 \quad \text{when } 0<r<b,$$
$$F'(r)=-\frac{4A}{\pi(\vartheta_1+\vartheta_2)}\left(2\sqrt{r^2-b^2}-b\arccos\frac{b}{r}\right), \quad \text{when } b<r<a. \tag{176}$$

Substituting (176) into (78), we find

$$\left.\begin{aligned} p(r)&=\frac{2A}{\pi^2(\vartheta_1+\vartheta_2)}\int_b^a\left(2\sqrt{s^2-b^2}-b\arccos\frac{b}{s}\right)\frac{ds}{\sqrt{s^2-r^2}} \quad \text{when } 0<r<b,\\ p(r)&=\frac{2A}{\pi^2(\vartheta_1+\vartheta_2)}\int_r^a\left(2\sqrt{s^2-b^2}-b\arccos\frac{b}{s}\right)\frac{ds}{\sqrt{s^2-r^2}} \quad \text{when } b<r<a.\end{aligned}\right\} \tag{177}$$

Assuming in (177)

$$s=\frac{b}{\cos\varphi},$$

we obtain

$$\left.\begin{aligned} p(r)&=\frac{2Ab}{\pi^2(\vartheta_1+\vartheta_2)}\int_0^{\varphi_0}\frac{(2\operatorname{tg}\varphi-\varphi)\operatorname{tg}\varphi\,d\varphi}{\sqrt{1-\frac{r^2}{b^2}\cos^2\varphi}} \quad \text{when } 0<r<b,\\ p(r)&=\frac{2Ab}{\pi^2(\vartheta_1+\vartheta_2)}\int_{\arccos\frac{b}{r}}^{\varphi_0}\frac{(2\operatorname{tg}\varphi-\varphi)\operatorname{tg}\varphi\,d\varphi}{\sqrt{1-\frac{r^2}{b^2}\cos^2\varphi}} \quad \text{when } b<r<a,\end{aligned}\right\} \tag{178}$$

since according to (169) arc cos $\frac{b}{a}=\varphi_0$.

Introducing the designation

$$\left.\begin{aligned} \psi(x)&=\int_0^{\varphi_0}\frac{(2\operatorname{tg}\varphi-\varphi)\operatorname{tg}\varphi\,d\varphi}{\sqrt{1-x^2\cos^2\varphi}} \quad \text{when } 0<x<1,\\ \psi(x)&=\int_{\arccos\frac{1}{x}}^{\varphi_0}\frac{(2\operatorname{tg}\varphi-\varphi)\operatorname{tg}\varphi\,d\varphi}{\sqrt{1-x^2\cos^2\varphi}} \quad \text{when } 1<x<\frac{1}{\cos\varphi_0},\end{aligned}\right\} \tag{179}$$

we will be able to give to formula (178) the form

$$p(r) = \frac{2Ab}{\pi^2(\vartheta_1+\vartheta_2)}\psi\left(\frac{r}{b}\right), \quad 0<r<a. \tag{180}$$

From (172) we find

$$\frac{2Ab}{\pi^2(\vartheta_1+\vartheta_2)} = \frac{3\cos^3\varphi_0}{3\sin\varphi_0+\sin^3\varphi_0-3\varphi_0\cos\varphi_0}\,\frac{P}{\pi b^2}. \tag{181}$$

Substituting (181) into (180), we obtain

$$p(r) = \frac{3\cos^3\varphi_0}{3\sin\varphi_0+\sin^3\varphi_0-3\varphi_0\cos\varphi_0}\,\psi\left(\frac{r}{b}\right)\frac{P}{\pi b^2}, \quad 0<r<a. \tag{182}$$

Formula (182) jointly with relations (179) determines the distribution of pressure in the region of contact. Figure 46 shows graphs of pressure $p(r)$, which correspond to different values of ratio $k = a/b$, i.e., different values of angle ϕ_0 appearing in formulas (182) and (179). Calculation of definite integrals appearing in formulas (179) was produced by the approximate Simpson formula.

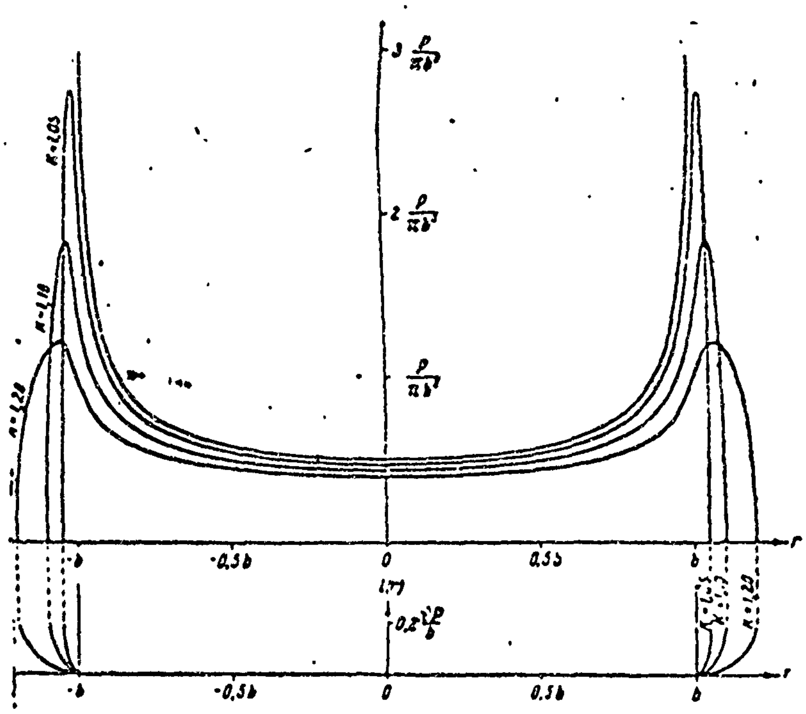

Fig. 46.

CHAPTER IV

GENERAL CASE OF THE CONTACT PROBLEM

§ 1. Potential of the Elliptic Disk

As we already saw above, the two-dimensional and axisymmetrical contact problems of the theory of elasticity lead to equations for which it is possible to plot general solutions in closed form. In the absence of radial symmetry three-dimensional contact problem of the theory of elasticity proves to be incomparably more complicated. Those solutions of it which we discuss in this book are based on certain properties of the potential of the elliptic disk, to the examination of which we now turn.

Let us construct a system of rectangular coordinates, x, y, z, combining plane xOy with the plane of the elliptic disk (Fig. 47). Then, according to determination of the Newtonian potential, potential $V(x, y, z)$ of the elliptic disk at point A with coordinates x, y, z will be equal to

$$V(x, y, z)=\iint_{\Sigma}\frac{p(x', y')\,dx'dy'}{R}, \qquad (1)$$

where $p(x', y')$ – density at point A' with coordinates x', y', 0, R – distance between points A and A', Σ – region of integration constituting the part of plane xOy occupied by the elliptic disk.

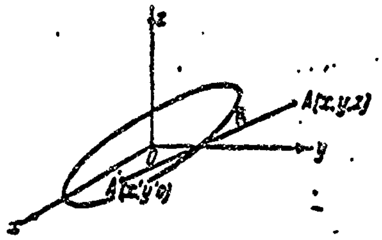

Fig. 47.

In particular, if point A lies on the surface of the disk, then

$$z=0, \quad R=\sqrt{(x-x')^2+(y-y')^2}$$

and according to (1)

$$V(x, y, 0)=\iint_{\Sigma}\frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}}. \tag{2}$$

If semiaxes of the ellipse, which limit the elliptic disk, are designated by a and b, then with the appropriate location of axes x and y the equation of this ellipse will have the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1. \tag{3}$$

As we show below, if density $p(x', y')$ has the form

$$p(x', y')=\frac{\sum_{m=0}^{n} a_m\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)^m}{\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}}, \tag{4}$$

where a_0, a_1, ..., a_n are constant coefficients, then the potential of the elliptic disk on its surface, determined by formula (2), is expressed by a polynomial in coordinates x and y of power $2n$. Based on this property of the potential of the elliptic disk are solutions of contact problems discussed in subsequent sections. Let us turn to proof of the indicated property of the potential of the elliptic disk.

Substituting (4) into (2), we find

$$V(x, y, 0) = \sum_{m=0}^{n} a_m f_m(x, y), \tag{5}$$

where

$$f_m(x, y) = \iint_S \left(1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right)^{m-\frac{1}{2}} \frac{dx'dy'}{\sqrt{(x-x')^2 + (y-y')^2}}, \tag{6}$$
$$m = 0, 1, \ldots, n.$$

Let us turn in the multiple integral (6) from rectangular coordinates x', y' to polar coordinates R, ϕ with the origin at point A with coordinates x, y (Fig. 48). As can be seen from Fig. 48,

$$\left.\begin{aligned} x' &= x + R\cos\varphi, \\ y' &= y + R\sin\varphi, \end{aligned}\right\} \tag{7}$$

instead of the area element $dx'dy'$ we will have the area element $d\sigma = RdRd\phi$, and formula (6) will take the form

$$f_m(x, y) = \int_0^{2\pi} d\varphi \cdot \int_0^{R_0(\varphi)} \left[1 - \frac{(x + R\cos\varphi)^2}{a^2} - \frac{(y + R\sin\varphi)^2}{b^2}\right]^{m-\frac{1}{2}} dR, \tag{8}$$

where $R_0(\phi)$ – distance between points A and A'' in Fig. 48. Since point A'' lies on the ellipse with semiaxes a and b, its coordinates x'', y'' must satisfy the relation

$$\frac{x''^2}{a^2} + \frac{y''^2}{b^2} = 1. \tag{9}$$

At the same time, as one can see from Fig. 48,

$$\left.\begin{aligned} x'' &= x + R_0\cos\varphi, \\ y'' &= y + R_0\sin\varphi. \end{aligned}\right\} \tag{10}$$

Substituting (10) into (9), we obtain the equation

$$\frac{1}{a^2}(x + R_0\cos\varphi)^2 + \frac{1}{b^2}(y + R_0\sin\varphi)^2 = 1,$$

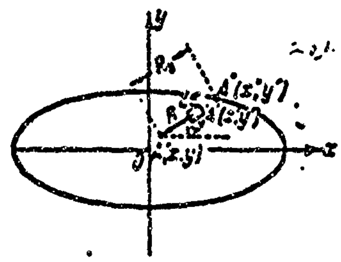

Fig. 48.

or

$$L(\varphi)R_0^2+2M(\varphi)R_0-N=0, \tag{11}$$

where

$$\left.\begin{aligned} L(\varphi)&=\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2},\\ M(\varphi)&=\frac{x\cos\varphi}{a^2}+\frac{y\sin\varphi}{b^2},\\ N&=1-\frac{x^2}{a^2}-\frac{y^2}{b^2}; \end{aligned}\right\} \tag{12}$$

Let us note that

$$N>0, \tag{13}$$

since the point with coordinates x, y lies inside the ellipse.

Solving equation (11), we obtain

$$R_0(\varphi)=\frac{-M(\varphi)\pm\sqrt{M^2(\varphi)+NL(\varphi)}}{L(\varphi)}. \tag{14}$$

As can be seen from (12),

$$L(\varphi)>0 \text{ when } 0<\varphi<2\pi. \tag{15}$$

In virtue of inequalities (13) and (15) we will have

$$\sqrt{M^2(\varphi)+NL(\varphi)}>|M(\varphi)|. \tag{16}$$

Thus, from the two solutions of the quadratic equation (11), determined by formula (14), one is always positive and the other negative. In order to obtain a positive solution of equation (11), which is of interest to us, one should take a plus sign before the radical in formula (14). Thus, finally we find

$$R_0(\varphi) = \frac{-M(\varphi) + \sqrt{M^2(\varphi) + NL(\varphi)}}{L(\varphi)}. \tag{17}$$

Formula (8) can be given the form

$$J_m(x, y) = \int_0^{2\pi} d\varphi \int_0^{R_0(\varphi)} \left[1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} - 2R\left(\frac{x\cos\varphi}{a^2} + \frac{y\sin\varphi}{b^2}\right) - R^2\left(\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2}\right)\right]^{m-\frac{1}{2}} dR = \int_0^{2\pi} d\varphi \int_0^{R_0(\varphi)} \left[N - 2RM(\varphi) - R^2L(\varphi)\right]^{m-\frac{1}{2}} dR,$$

according to designations (12), or

$$\begin{aligned} J_m(x, y) &= \\ &= \int_0^{2\pi} d\varphi \int_0^{R_0(\varphi)} L^{\frac{1}{2}-m}(\varphi)\left[NL(\varphi) - 2RL(\varphi)M(\varphi) - R^2L^2(\varphi)\right]^{m-\frac{1}{2}} dR = \\ &= \int_0^{2\pi} d\varphi \int_0^{R_0(\varphi)} L^{\frac{1}{2}-m}(\varphi)\left\{M^2(\varphi) + NL(\varphi) - \left[M(\varphi) + RL(\varphi)\right]^2\right\}^{m-\frac{1}{2}} dR = \\ &= \int_0^{2\pi} d\varphi \int_0^{R_0(\varphi)} \left[\frac{M^2(\varphi)}{L(\varphi)} + N\right]^{m-\frac{1}{2}} \left\{1 - \left[\frac{M(\varphi) + RL(\varphi)}{\sqrt{M^2(\varphi) + NL(\varphi)}}\right]^2\right\}^{m-\frac{1}{2}} dR. \end{aligned} \tag{18}$$

(According to inequalities (13) and (15) the subradical expression in formula (18) is essentially positive.)

Since $0 \leqslant R < R_0(\varphi)$, and function $L(\phi)$ according to (15) is positive, we have

$$\frac{M(\varphi)}{\sqrt{M^2(\varphi) + NL(\varphi)}} < \frac{M(\varphi) + RL(\varphi)}{\sqrt{M^2(\varphi) + NL(\varphi)}} < \frac{M(\varphi) + R_0(\varphi)L(\varphi)}{\sqrt{M^2(\varphi) + NL(\varphi)}}. \tag{19}$$

But according to (16)

$$\frac{M(\varphi)}{\sqrt{M^2(\varphi)+NL(\varphi)}} > -1, \tag{20}$$

and according to (17)

$$\frac{M(\varphi)+R_0(\varphi)L(\varphi)}{\sqrt{M^2(\varphi)+NL(\varphi)}} = 1. \tag{21}$$

From relations (19), (20) and (21) we get

$$-1 < \frac{M(\varphi)+RL(\varphi)}{\sqrt{M^2(\varphi)+NL(\varphi)}} < 1, \tag{22}$$

and, consequently, instead of the variable of integration R it is possible to take a new variable ϑ, assuming

$$\cos\vartheta = \frac{M(\varphi)+RL(\varphi)}{\sqrt{M^2(\varphi)+NL(\varphi)}} \quad (0<\vartheta<\pi). \tag{23}$$

From (23) we find

$$-\sin\vartheta\, d\vartheta = \frac{L(\varphi)}{\sqrt{M^2(\varphi)+NL(\varphi)}}\, dR, \tag{24}$$

$$dR = -\sqrt{\frac{M^2(\varphi)}{L(\varphi)}+N}\,\frac{\sin\vartheta\, d\vartheta}{\sqrt{L(\varphi)}}.$$

From relations (23) and (21) it follows that

$$\vartheta = 0 \text{ when } R = R_0(\varphi). \tag{25}$$

Producing in (18) replacement of the variable of integration R by ϑ, according to (23), (24) and (25) we will have

$$f_m(x,y) = \int_0^{2\pi} d\varphi \int_0^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^m\vartheta\, d\vartheta, \tag{26}$$

where $\theta(\phi)$ is the value which takes the variable of integration ϑ, when $R = 0$. From (23) we find

$$\cos\theta(\varphi) = \frac{M(\varphi)}{\sqrt{M^2(\varphi) + NL(\varphi)}}. \tag{27}$$

Assuming

$$\varphi = \pi + \psi, \tag{28}$$

we find

$$\int_{\pi}^{2\pi} d\varphi \int_{0}^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}} \left[\frac{M^2(\varphi)}{L(\varphi)} + N\right]^m \sin^{2m}\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\psi \int_{0}^{\theta(\pi+\psi)} \frac{1}{\sqrt{L(\pi+\psi)}} \left[\frac{M^2(\pi-\psi)}{L(\pi+\psi)} + N\right]^m \sin^{2m}\theta\, d\theta. \tag{29}$$

Substituting (28) into (12), we obtain

$$L(\pi+\psi) = L(\psi), \quad M(\pi+\psi) = -M(\psi). \tag{30}$$

Substituting (28) into (27) and taking into account (30), we have

$$\cos\theta(\pi+\psi) = -\cos\theta(\psi),$$

whence

$$\theta(\pi+\psi) = \pi - \theta(\psi). \tag{31}$$

Substituting (30) and (31) into (29), we find

$$\int_{\pi}^{2\pi} d\varphi \int_{0}^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}} \left[\frac{M^2(\varphi)}{L(\varphi)} + N\right]^m \sin^{2m}\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\psi \int_{0}^{\pi-\theta(\psi)} \frac{1}{\sqrt{L(\psi)}} \left[\frac{M^2(\psi)}{L(\psi)} + N\right]^m \sin^{2m}\theta\, d\theta. \tag{32}$$

Assuming in (32) $\theta = \pi - \theta'$, we obtain

$$\int_{\pi}^{2\pi} d\varphi \int_{0}^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^{2m}\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\psi \int_{\theta(\psi)}^{\pi} \frac{1}{\sqrt{L(\psi)}}\left[\frac{M^2(\psi)}{L(\psi)}+N\right]^m \sin^{2m}\theta' d\theta'. \qquad (33)$$

Using relation (33), we will be able to give to formula (26) the form

$$f_m(x, y) = \int_{0}^{\pi} d\varphi \int_{0}^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^{2m}\theta\, d\theta +$$

$$+ \int_{\pi}^{2\pi} d\varphi \int_{0}^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^{2m}\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\varphi \int_{0}^{\theta(\varphi)} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^{2m}\theta\, d\theta +$$

$$+ \int_{0}^{\pi} d\varphi \int_{\theta(\varphi)}^{\pi} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^{2m}\theta\, d\theta =$$

$$= \int_{0}^{\pi} d\varphi \int_{0}^{\pi} \frac{1}{\sqrt{L(\varphi)}}\left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \sin^{2m}\theta\, d\theta,$$

or finally

$$f_m(x, y) = c_m \int_{0}^{\pi} \left[\frac{M^2(\varphi)}{L(\varphi)}+N\right]^m \frac{d\varphi}{\sqrt{L(\varphi)}}, \qquad (34)$$

where

$$c_m = \int_{0}^{\pi} \sin^{2m}\theta\, d\theta. \qquad (35)$$

Substituting (12) into (34), we obtain

$$f_m(x, y) = c_m ab \int_{0}^{\pi} \left[\frac{(xb^2\cos\varphi + ya^2\sin\varphi)^2}{a^2b^2(a^2\sin^2\varphi + b^2\cos^2\varphi)} + 1 - \frac{b^2x^2+a^2y^2}{a^2b^2}\right]^m \times$$

$$\times \frac{d\varphi}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}} =$$

$$= c_m ab \int_{0}^{\pi} \left[1 - \frac{(b^2x^2+a^2y^2)(a^2\sin^2\varphi+b^2\cos^2\varphi)-(xb^2\cos\varphi+ya^2\sin\varphi)^2}{a^2b^2(a^2\sin^2\varphi+b^2\cos^2\varphi)}\right]^m \times$$

$$\times \frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}} =$$

$$= c_m ab \int_0^\pi \left[1-\frac{x^2a^2b^2\sin^2\varphi+y^2a^2b^2\cos^2\varphi-2xya^2b^2\sin\varphi\cos\varphi}{a^2b^2(a^2\sin^2\varphi+b^2\cos^2\varphi)}\right]^m \times$$

$$\times \frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}},$$

or finally

$$I_m(x,y)=c_m ab\int_0^\pi\left[1-\frac{(x\sin\varphi-y\cos\varphi)^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}\right]^m\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}. \qquad (36)$$

Substituting (36) into (5), we find

$$V(x,y,0)=$$

$$=ab\int_0^\pi\sum_{m=0}^{n}a_mc_m\left[1-\frac{(x\sin\varphi-y\cos\varphi)^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}\right]^m\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}. \qquad (37)$$

Let us now turn to the calculation of coefficients c_m in formula (37). Fulfilling in formula (35) partial integration, we find

$$c_m=-\int_0^\pi\sin^{2m-1}\theta\, d(\cos\theta)=-\sin^{2m-1}\theta\cos\theta\Big|_0^\pi+$$

$$+(2m-1)\int_0^\pi\sin^{2m-2}\theta\cos^2\theta\, d\theta=(2m-1)\left(\int_0^\pi\sin^{2m-2}\theta\, d\theta-\right.$$

$$\left.-\int_0^\pi\sin^{2m}\theta\, d\theta\right)=(2m-1)c_{m-1}-(2m-1)c_m,$$

whence

$$c_m=\frac{2m-1}{2m}c_{m-1}. \qquad (38)$$

Assuming in (38) $m-1$, $m-2$, etc., instead of m, we obtain the relations

$$c_{m-1}=\frac{2m-3}{2m-2}c_{m-2}, \qquad (39)$$

$$c_{m-2}=\frac{2m-5}{2m-4}c_{m-3}, \qquad (40)$$

etc. Substituting (39) into (38), we find

$$c_m = \frac{(2m-1)(2m-3)}{2m(2m-2)} c_{m-2}. \tag{41}$$

Substituting (40) into (41), we obtain

$$c_m = \frac{(2m-1)(2m-3)(2m-5)}{2m(2m-2)(2m-4)} c_{m-3}. \tag{42}$$

Continuing this process, we arrive at the formula

$$c_m = \frac{(2m-1)(2m-3)\ldots 3\cdot 1}{2m(2m-2)\ldots 4\cdot 2} c_0. \tag{43}$$

Setting $m = 0$ in (35) we find

$$c_0 = \int_0^\pi d\vartheta = \pi. \tag{44}$$

Substituting (44) into (43), we obtain

$$c_m = \frac{1\cdot 3\cdot 5\ldots(2m-3)(2m-1)}{2\cdot 4\cdot 6\ldots(2m-2)\,2m}\pi, \qquad m = 1, 2, \ldots \tag{45}$$

Substituting (44) and (45) into (37), we find

$$V(x, y, 0) = \pi ab \int_0^\pi \left\{a_0 + \sum_{m=1}^{n} \frac{1\cdot 3\cdot 5\ldots(2m-3)(2m-1)}{2\cdot 4\cdot 6\ldots(2m-2)\,2m} a_m \times \right.$$
$$\left. \times \left[1 - \frac{(x\sin\varphi - y\cos\varphi)^2}{a^2\sin^2\varphi + b^2\cos^2\varphi}\right]^m\right\} \times \frac{d\varphi}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}. \tag{46}$$

Thus, the potential of the elliptic disk is expressed on the surface of this disk by formula (46), if density p is expressed by formula (4). Expression (46) obtained by us for the unknown potential indeed represents with respect to variables x and y the polynomial of the $2n$ power, which was required to be shown.

§ 2. Pressure of the Elliptic Stamp on the Elastic Half-Space

In Chapter III we examined the problem on the pressure of the circular cylindrical stamp on the elastic half-space. In this chapter we examine the problem on the pressure on the elastic half-space of a rigid cylindrical stamp with elliptic cross section.

Let us designate by a and b (let us agree that $a \leq b$) the semiaxes of the ellipse limiting the base of the stamp, and let us plot the system of rectangular coordinates x, y, z in such a way that the equation of the curve limiting the region of contact of the stamp with the half-space has the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \qquad z=0.$$

and that the elastic half-space covers the negative semiaxis z (Fig. 49). Let us designate further by $p(x, y)$ the normal pressure appearing under the stamp at the point with coordinates x, y (the base of the stamp will be considered ideally smooth). Under the action of the mentioned pressure the point of the surface of the elastic half-space with coordinates x, y should accomplish an elastic displacement with projection u_z on axis z equal to (see formula (69) in Chapter III)

$$u_z=-\theta\iint_{\Sigma}\frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}}, \qquad (47)$$

where $\theta=\frac{1-\mu^2}{\pi E}$, E – elastic modulus, μ – Poisson's ratio of the elastic medium, Σ – region in which pressure $p(x, y)$ acts, in our case the region limited by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Let us designate further by α the forward displacement which the stamp accomplishes in the direction of the negative semiaxis z

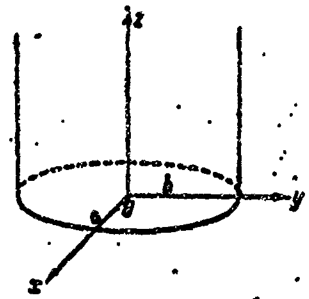

Fig. 49.

with compression. Each point of the elastic half-space found in contact with the stamp should with compression undergo elastic displacement in the direction of the negative semiaxis z equal to α, i.e., in the whole region of contact the condition

$$u_z = -\alpha, \tag{48}$$

should be fulfilled. By comparing relations (47) and (48), we find that in the region of contact the condition

$$\iint_{\Sigma} \frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}} = \frac{\alpha}{\theta}. \tag{49}$$

should be fulfilled. The expression standing in the left side of relation (49) at the point with coordinates x, y, 0 determines the potential $V(x, y, 0)$ of the elliptic disk with density p (see formula (2)), and in the right side of equality (49) there is the constant ratio $\frac{\alpha}{\theta}$.

Thus, the problem of detecting pressure p under the stamp is equivalent to the detecting of that density p at which the potential of elliptic disk maintains a constant value on its surface. In the preceding section we showed that if density p is determined by formula (4), the potential of the disk $V(x, y, 0)$ is determined by formula (46), constitutes a polynomial of the $2n$ power with respect to variables x and y and, in particular, with $n = 0$ maintains the constant value. Assuming in (4) and (46) $n = 0$ we find

$$p(x', y') = \frac{a_0}{\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}}, \tag{50}$$

$$V(x, y, 0) = \pi a b a_0 \int_0^{\pi} \frac{d\varphi}{\sqrt{a^2 \sin^2 \varphi + b^2 \cos^2 \varphi}}. \tag{51}$$

Thus, if for pressure $p(x', y')$ we take the expression (50), then the multiple integral standing in the left side of relation (49), will maintain the constant value determined by formula (51). Thus, in order that in the region of contact condition (49) be fulfilled, it is sufficient that this constant value be equal to $\frac{\alpha}{\theta}$. Hence we obtain the relation

$$\pi a b a_0 \int_0^{\pi} \frac{d\varphi}{\sqrt{a^2 \sin^2 \varphi + b^2 \sin^2 \varphi}} = \frac{\alpha}{\theta}, \tag{52}$$

which connects coefficient a_0, appearing in expression (50) obtained for pressure p, and the approach of the stamp with the elastic medium α.

Let us designate by P the force pressing the stamp to the elastic half-space. This force should be balanced by the reaction of the elastic half-space. Consequently, integrating pressure p with respect to the whole region of contact, we should obtain force P:

$$\iint_{\Sigma} p(x', y')\, dx'\, dy' = P. \tag{53}$$

Substituting (50) into (53), we obtain the relation

$$a_0 \iint_{\Sigma} \frac{dx'\, dy'}{\sqrt{1 - \frac{x'^2}{a^2} - \frac{y'^2}{b^2}}} = P. \tag{54}$$

Formulas (54) and (52) determine the constant a_0 and approach α. In order to calculate the multiple integral entering into formula (54), let us cross over to it from the rectangular coordinates x', y' to polar coordinates r, ϕ, assuming

$$x' = r \cos \varphi, \quad y' = r \sin \varphi.$$

Let us find

$$\iint_{S} \frac{dx'\,dy'}{\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}} = \int_0^{2\pi} d\varphi \int_0^{r_0(\varphi)} \frac{r\,dr}{\sqrt{1-\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)r^2}}. \qquad (55)$$

The limit of integration $r_0(\phi)$ is determined by the condition that the point with rectangular coordinates

$$x' = r_0\cos\varphi, \quad y' = r_0\sin\varphi,$$

should lie on the ellipse

$$\frac{x'^2}{a^2}+\frac{y'^2}{b^2}=1.$$

Hence

$$r_0^2(\varphi)\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)=1. \qquad (56)$$

Fulfilling in (55) integration with respect to r and taking into consideration relation (56), we find

$$\begin{aligned}\iint_{S} \frac{dx'\,dy'}{\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}} &= \\ &= -\int_0^{2\pi} \sqrt{1-\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)r^2}\,\Bigg|_{r=0}^{r=r_0(\varphi)} \frac{d\varphi}{\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}} = \\ &= a^2b^2\int_0^{2\pi}\frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = 2a^2b^2\int_0^{\pi}\frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi},\end{aligned} \qquad (57)$$

since in the last definite integral the integrand has the period π. Assuming further $\varphi=\pi-\psi$, we find

$$\int_{\frac{\pi}{2}}^{\pi}\frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = \int_0^{\frac{\pi}{2}}\frac{d\psi}{a^2\sin^2\psi+b^2\cos^2\psi},$$

whence

$$\int_0^{\pi} \frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = \int_0^{\frac{\pi}{2}} \frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} + \int_{\frac{\pi}{2}}^{\pi} \frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} =$$

$$= 2\int_0^{\frac{\pi}{2}} \frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi}. \tag{58}$$

Let us introduce in the last definite integral instead of the variable of integration ϕ the new variable of integration t, setting

$$\operatorname{tg}\varphi = \frac{b}{a}t.$$

Let us find

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = \int_0^{\frac{\pi}{2}} \frac{\sec^2\varphi\,d\varphi}{a^2\operatorname{tg}^2\varphi+b^2} = \int_0^{\frac{\pi}{2}} \frac{d(\operatorname{tg}\varphi)}{a^2\operatorname{tg}^2\varphi+b^2} =$$

$$= \int_0^{\infty} \frac{\frac{b}{a}dt}{b^2t^2+b^2} = \frac{1}{ab}\int_0^{\infty} \frac{dt}{1+t^2} = \frac{1}{ab}\operatorname{arc\,tg} t\Big|_0^{\infty} = \frac{\pi}{2ab}. \tag{59}$$

Substituting (59) into (58), we obtain

$$\int_0^{\pi} \frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = \frac{\pi}{ab}. \tag{60}$$

Substituting (60) into (57), we find

$$\iint_{\Sigma} \frac{dx'\,dy'}{\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}} = 2\pi ab, \tag{61}$$

Using formula (61), we obtain from relation (54) the value of the constant a_0:

$$a_0 = \frac{P}{2\pi ab}. \tag{62}$$

Substituting (62) into (50), we find the final expression for pressure $p(x, y)$ in the region of contact

$$p(x, y)=\frac{P}{2\pi ab\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}}. \tag{63}$$

Substituting (62) into (52), let us determine the displacement of the stamp with compression α:

$$\alpha=\frac{P\vartheta}{2}\int_0^{\pi}\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}. \tag{64}$$

Formula (64) can be given the form

$$\alpha=\frac{P\vartheta}{2}\int_0^{\pi}\frac{d\varphi}{\sqrt{b^2+(a^2-b^2)\sin^2\varphi}}=\frac{P\vartheta}{2b}\int_0^{\pi}\frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}}, \tag{65}$$

where

$$e=\sqrt{1-\frac{a^2}{b^2}} \tag{66}$$

is the eccentricity of the ellipse (we agreed that $a \leq b$). Since the integrand in (65) remains constant with replacement of ϕ by $\pi - \phi$,

$$\int_0^{\pi}\frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}}=2\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}},$$

and formula (65) can be given the form

$$\alpha=\frac{P\vartheta}{b}\int_0^{\frac{\pi}{2}}\frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}}. \tag{67}$$

As we already repeatedly noted, the definite integral

$$F(x, k) = \int_0^x \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} \tag{68}$$

is called the elliptical integral of the first kind with modulus k. In the case when the upper limit x is equal to unity, the elliptic integral is called complete, and we denote

$$F(1, k) = K(k). \tag{68'}$$

Setting $\sin\varphi = x$,

$$d\varphi = \frac{dx}{\cos\varphi} = \frac{dx}{\sqrt{1-x^2}};$$

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}} = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-e^2x^2)}} = K(e). \tag{69}$$

Formula (67) can be thus given the form

$$\alpha = \frac{P\vartheta}{b} K(e). \tag{70}$$

Displacement of the stamp with compression α is expressed in terms of the complete elliptic integral of the first kind with the modulus equal to the eccentricity of the ellipse limiting the base of the stamp.

Formulas (63) and (70) completely solve the contact problem examined in this chapter by determining the pressure under the stamp $p(x, y)$ and approach of the stamp with the elastic medium α. As can be seen from formula (63), when the point with coordinates x, y approaches toward the ellipse, which limits the base of stamp, the denominator in the expression determining pressure $p(x, y)$ approaches zero, and pressure $p(x, y)$ increases without limit. In the contact problem examined by us the section of the stamp by a plane passing through the z axis has at the base of the stamp right angles (see Fig. 49). In reality for any real stamp such a section will have at the base of the stamp a large but limited curvature. In this case although pressure $p(x, y)$ can reach at edges of the base of the stamp

great values, nevertheless it remains limited in the whole region of contact. By examining the two-dimensional and axisymmetrical contact problems, we examined this question in detail. In the absence of radial symmetry this question in the three-dimensional contact problem leads, unfortunately, to great mathematical difficulties, and we do not discuss it in greater detail.

§ 3. Compression of Two Elastic Bodies Initially Touching at a Point

In Chapter III we examined the problem on the compression of two elastic bodies initially touching at a point for that case when both compressible bodies have a common axis of radial symmetry. In this chapter we examine the general case of this problem, assuming that the compressible bodies have an arbitrary configuration.

Let us construct a system of rectangular coordinates x, y, z, disposing the origin of the coordinates at the point of initial contact of compressible bodies and combining the plane xOy with the common tangent plane to surfaces of compressible bodies at the point of their contact (Fig. 50). Let us assume that

$$\left.\begin{aligned} z &= f_1(x, y), \\ z &= -f_2(x, y) \end{aligned}\right\} \tag{71}$$

are equations of surfaces limiting the compressible bodies. Let us assume that, further, A_1 and A_2 are two points of these surfaces touching with compression; A_1B_1 and A_2B_2 – elastic displacements of these points (see Fig. 50). Points B_1 and B_2 are combined with compression due to forward displacements of the compressible bodies causing the approach of them, which we will designate by α. We will subsequently assume that resultants of compressing forces lie on the z axis, and the indicated approach of the compressible bodies with compression is carried out along the z axis. Under these assumptions segment B_1B_2 on Fig. 50 should be parallel to the z axis. Let us designate by z_1 and z_2 coordinates z of points A_1 and A_2.

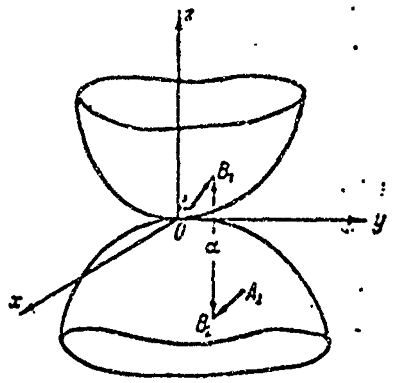

Fig. 50.

Then the z coordinates of points B_1 and B_2 will be equal to $z_1 + u_{1z}$ and $z_2 + u_{2z}$, where u_{1z} and u_{2z} are projections of elastic displacements of points A_1 and A_2 on the z axis. The distance between points B_1 and B_2 thus equal to $z_1 + u_{1z} - (z_2 + u_{2z})$. On the other hand, this same distance is equal to the approach of the compressible bodies α. Consequently, for every pair of points touching with compression, we should observe the condition

$$z_1 + u_{1z} - (z_2 + u_{2z}) = \alpha. \qquad (72)$$

Let us designate, further, by x, y the corresponding coordinates of points B_1 and B_2. In view of the smallness of elastic displacements it is possible approximately to assume that

$$z_1 = f_1(x, y), \quad z_2 = -f_2(x, y), \qquad (73)$$

in accordance with equations of surfaces of compressible bodies (71). Substituting (73) into (72), we obtain

$$u_{1z} - u_{2z} = \alpha - f_1(x, y) - f_2(x, y). \qquad (74)$$

Let us designate, further, by Σ the projection of the region of contact on plane xOy and by $p(x, y)$ the normal pressure at the point of contact with coordinates x, y. We will consider subsequently surfaces of compressible bodies to be perfectly smooth and assume the displacement u_{2z} approximately equal to that displacement which is accomplished in the direction of the z axis the point of elastic

half-space $z < 0$ with coordinates x, y, 0 under the effect of normal pressure $p(x, y)$ effective in region Σ. Then

$$u_{z2} = -\vartheta_2 \iint\limits_{\Sigma} \frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}} \tag{75}$$

(see formula (47)), where $\vartheta_2 = \frac{1-\mu_2^2}{\pi E_2}$, E_2 — elastic modulus, μ_2 — Poisson coefficient for the second of the compressible bodies.

Under similar assumptions we will have

$$u_{z1} = \vartheta_1 \iint\limits_{\Sigma} \frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}}, \tag{76}$$

where $\vartheta_1 = \frac{1-\mu_1^2}{\pi E_1}$, E_1 — elastic modulus, and μ_1 — Poisson coefficient for the first of the compressible bodies.

Substituting (75) and (76) into (74), let us find that in region Σ there should be fulfilled the condition

$$\iint\limits_{\Sigma} \frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}} = \frac{\alpha - f_1(x, y) - f_2(x, y)}{\vartheta_1+\vartheta_2}. \tag{77}$$

Since according to condition the origin of the coordinates is the regular point for surfaces of compressible bodies, functions $f_1(x, y)$ and $f_2(x, y)$ in equations of these surfaces (71) can be expanded in power series in neighborhood of the origin of coordinates

$$\left.\begin{aligned} f_1(x, y) &= a_0 + a_1 x + a_2 y + a_{11} x^2 + a_{12} xy + a_{22} y^2 + \ldots, \\ f_2(x, y) &= b_0 + b_1 x + b_2 y + b_{11} x^2 + b_{12} xy + b_{22} y^2 + \ldots \end{aligned}\right\} \tag{78}$$

Since surfaces of compressible bodies pass through the origin of the coordinates and planes xOy touch, we will have

$$a_0 = a_1 = a_2 = b_0 = b_1 = b_2 = 0. \tag{79}$$

Taking into account (79), we find from (78)

$$f_1(x, y)+f_2(x, y)=(a_{11}+b_{11})x^2+ \\ +(a_{12}+b_{12})xy+(a_{22}+b_{22})y^2+\dots \qquad (80)$$

The direction of coordinate axes x and y have thus far been left arbitrary. Let us now orient these coordinate axes in such a way so that in the expansion of (80) the coefficient at xy turns into zero and inequality $a_{11}+b_{11}\geqslant a_{22}+b_{22}$, takes place, i.e., so that conditions

$$a_{12}+b_{12}=0, \quad a_{11}+b_{11}>a_{22}+b_{22}. \qquad (81)$$

were fulfilled. We will assume that here not one of coefficients a_{11} + b_{11} and a_{22} + b_{22} turns into zero. Then, if we disregard in the expansion of (80) smalls of higher orders (proceeding from the smallness of the region of contact) in region Σ we will have

$$f_1(x, y)+f_2(x, y)=(a_{11}+b_{11})x^2+(a_{22}+b_{22})y^2. \qquad (82)$$

Substituting (82) into (77), let us find that in region Σ condition

$$\iint\limits_{\Sigma}\frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}}=\frac{\alpha-Ax^2-By^2}{\vartheta_1+\vartheta_2}, \qquad (83)$$

should be fulfilled where

$$A=a_{11}+b_{11}, \quad B=a_{22}+b_{22} \quad (A>B). \qquad (84)$$

Integrating the pressure $p(x'y')$ with respect to region Σ, we should obtain the resultant of external compressing forces acting on each of the compressible bodies. Designating this resultant by P, we will obtain the condition

$$\iint\limits_{\Sigma}p(x', y')\,dx'\,dy'=P. \qquad (85)$$

[illegible] r to solve the examined contact problem, it is necessary [illegible] Σ, pressure $p(x, y)$ and approach α by proceeding [illegible] 83) and (85).

The expression standing in the left side of relation (83) determines at the point with coordinates x, y, 0 the potential of the disk with density p, which cover in plane xOy the region Σ. In the right side of relation (84) there is a polynomial of the second power in x and y. As we showed in § 1, if the disk is limited by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \tag{86}$$

and density p is determined by formula (4), the potential of disk $V(x, y, 0)$ is a polynomial of the $2n$ power with respect to variables x and y determined by formula (46). Assuming in formulas (4) and (46) $n = 1$, we find that the integral standing in the left side of relation (84) will be in region Σ equal to

$$\iint_{\Sigma}\frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}}= \\ =\frac{\pi}{2}aba_1\int_0^{\pi}\left[1-\frac{(x\sin\varphi-y\cos\varphi)^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}\right]\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}, \tag{87}$$

if as region Σ we take the part of plane xOy limited by ellipse (86), and for pressure $p(x', y')$ we take expression

$$p(x', y')=a_1\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}. \tag{88}$$

Coefficient a_0 in formulas (4) and (46) were taken equal to zero in order for pressure $p(x', y')$ to obtain the expression limited in the whole region Σ.

Relation (87) can be given the form

$$\iint_{\Sigma}\frac{p(x', y')\,dx'\,dy'}{\sqrt{(x-x')^2+(y-y')^2}}=J_0-J_1x^2+J_2xy-J_3y^2, \tag{89}$$

where

$$J_0 = \frac{\pi}{2} aba_1 \int_0^{\pi} \frac{d\varphi}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}, \tag{90}$$

$$J_1 = \frac{\pi}{2} aba_1 \int_0^{\pi} \frac{\sin^2\varphi\, d\varphi}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2}}, \tag{91}$$

$$J_2 = \pi aba_1 \int_0^{\pi} \frac{\sin\varphi\cos\varphi\, d\varphi}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2}}, \tag{92}$$

$$J_3 = \frac{\pi}{2} aba_1 \int_0^{\pi} \frac{\cos^2\varphi\, d\varphi}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2}}. \tag{93}$$

As we already showed in the preceding chapter,

$$\int_0^{\pi} \frac{d\varphi}{\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}} = \frac{2}{b} K(e) \tag{94}$$

(compare formulas (64) and (70)), where

$$e = \sqrt{1 - \frac{a^2}{b^2}} \tag{95}$$

is the eccentricity of the ellipse limiting the region Σ (on the assumption that $a \leq b$,

$$K(e) = \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - e^2\sin^2\varphi}} \tag{96}$$

is the total elliptic integral of the first kind with modulus e. Substituting (94) into (90), we find

$$J_0 = \pi aa_1 K(e). \tag{97}$$

Formula (91) can be given the form

$$\begin{aligned} J_1 &= \frac{\pi}{2} aba_1 \int_0^{\pi} \frac{\sin^2\varphi\, d\varphi}{[b^2 - (b^2 - a^2)\sin^2\varphi]^{3/2}} = \frac{\pi aa_1}{2b^2} \int_0^{\pi} \frac{\sin^2\varphi\, d\varphi}{(1 - e^2\sin^2\varphi)^{3/2}} = \\ &= \frac{\pi aa_1}{2b^2e^2}\left[\int_0^{\pi} \frac{d\varphi}{(1 - e^2\sin^2\varphi)^{3/2}} - \int_0^{\pi} \frac{d\varphi}{\sqrt{1 - e^2\sin^2\varphi}}\right] = \\ &= \frac{\pi aa_1}{b^2e^2}\left[\int_0^{\pi/2} \frac{d\varphi}{(1 - e^2\sin^2\varphi)^{3/2}} - \int_0^{\pi/2} \frac{d\varphi}{\sqrt{1 - e^2\sin^2\varphi}}\right], \end{aligned} \tag{98}$$

since in these definite integrals the integrand remains constant with replacement of ϕ by $\pi - \phi$. In order to convert formula (98), let us use the identity

$$\frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{\sqrt{1-e^2\sin^2\varphi}}\right)=\frac{1-2\sin^2\varphi}{\sqrt{1-e^2\sin^2\varphi}}+\frac{e^2\sin^2\varphi(1-\sin^2\varphi)}{(1-e^2\sin^2\varphi)^{3/2}}=$$
$$=\frac{1-2\sin^2\varphi+e^2\sin^4\varphi}{(1-e^2\sin^2\varphi)^{3/2}}=\frac{e^2-1+(1-e^2\sin^2\varphi)^2}{e^2(1-e^2\sin^2\varphi)^{3/2}}=$$
$$=\frac{1}{e^2}\sqrt{1-e^2\sin^2\varphi}-\frac{1-e^2}{e^2}\frac{1}{(1-e^2\sin^2\varphi)^{3/2}}. \qquad (99)$$

Integrating both sides of identity (99) with respect to ϕ within limits of 0 to $\pi/2$, we find

$$0=\frac{1}{e^2}\int_0^{\pi/2}\sqrt{1-e^2\sin^2\varphi}\,d\varphi-\frac{1-e^2}{e^2}\int_0^{\pi/2}\frac{d\varphi}{(1-e^2\sin^2\varphi)^{3/2}},$$

whence

$$\int_0^{\pi/2}\frac{d\varphi}{(1-e^2\sin^2\varphi)^{3/2}}=\frac{1}{1-e^2}\int_0^{\pi/2}\sqrt{1-e^2\sin^2\varphi}\,d\varphi. \qquad (100)$$

The definite integral appearing in the right side of formula (100) is called the total elliptic integral of the second kind with modulus e and is designated by $E(e)$:

$$E(e)=\int_0^{\pi/2}\sqrt{1-e^2\sin^2\varphi}\,d\varphi. \qquad (101)$$

Thus, formula (100) can be given the form

$$\int_0^{\pi/2}\frac{d\varphi}{(1-e^2\sin^2\varphi)^{3/2}}=\frac{E(e)}{1-e^2}. \qquad (102)$$

According to (96) and (102) formula (98) can be presented in the form

$$J_1=\frac{\pi a a_1}{b^2e^2(1-e^2)}[E(e)-(1-e^2)K(e)]$$

or according to (95)

$$J_1 = \frac{\pi a_1}{a e^2}[E(e) - (1-e^2)K(e)]. \tag{103}$$

Let us turn to the calculation of the definite integral J_2 (see formula (92)). Let us find

$$\int_0^\pi \frac{\sin\varphi\cos\varphi\, d\varphi}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} = \frac{1}{(b^2-a^2)\sqrt{a^2\sin^2\varphi + b^2\cos^2\varphi}}\Bigg|_{\varphi=0}^{\varphi=\pi} = 0,$$

whence

$$J_2 = 0. \tag{104}$$

From formulas (91) and (93) we find

$$J_1 + J_3 = \frac{\pi}{2} a b a_1 \int_0^\pi \frac{d\varphi}{(a^2\sin^2\varphi + b^2\cos^2\varphi)^{3/2}} = \frac{\pi a a_1}{2b^2}\int_0^\pi \frac{d\varphi}{(1-e^2\sin^2\varphi)^{3/2}} =$$
$$= \frac{\pi a a_1}{b^2}\int_0^{\pi/2} \frac{d\varphi}{(1-e^2\sin^2\varphi)^{3/2}} = \frac{\pi a a_1}{b^2(1-e^2)} E(e)$$

according to (102), or

$$J_1 + J_3 = \frac{\pi a_1}{a} E(e) \tag{105}$$

in accordance with (95). Substituting value J_1 from (103) into (105) we find

$$J_3 = \frac{\pi a_1}{a e^2}[e^2 E(e) - E(e) + (1-e^2)K(e)],$$

or finally:

$$J_3 = \frac{\pi a_1 (1-e^2)}{a e^2}[K(e) - E(e)]. \tag{106}$$

Substituting (97), (103), (104) and (106) into (89), we find

$$\iint_{\Sigma} \frac{p(x', y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} = \pi a_1 \Big\{ aK(e) - \frac{1}{ae^2}[E(e)-(1-e^2)K(e)]x^2 - \\ - \frac{1-e^2}{ae^2}[K(e)-E(e)]y^2 \Big\}. \qquad (107)$$

Thus, if for pressure $p(x', y')$ we take expression (sic) the multiple integral standing in the left side of relation (sic) will be determined by formula (107). So that here condition (83) is fulfilled, it is sufficient that the polynomials standing in right sides of relations (83) and (107) would be identically equal to each other. Comparing the coefficients of these polynomials with each other, we will obtain the conditions

$$\pi a a_1 K(e) = \frac{a}{\vartheta_1+\vartheta_2}, \qquad (108)$$

$$\frac{\pi a_1}{ae^2}[E(e)-(1-e^2)K(e)] = \frac{A}{\vartheta_1+\vartheta_2}, \qquad (109)$$

$$\frac{\pi a_1}{ae^2}(1-e^2)[K(e)-E(e)] = \frac{B}{\vartheta_1+\vartheta_2}. \qquad (110)$$

Substituting (88) into (85), we obtain the additional condition

$$a_1 \iint_{\Sigma} \sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\,dx'dy' = P. \qquad (111)$$

In order to calculate the multiple integral entering into formula (111) let us turn in it from the rectangular coordinates x', y' to polar coordinates r, ϕ, assuming

$$x' = r\cos\varphi, \quad y' = r\sin\varphi.$$

Let us find

$$\iint_{\Sigma} \sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\,dx'dy' = \\ = \int_0^{2\pi} d\varphi \int_0^{r_0(\varphi)} \sqrt{1-\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)r^2}\, r\,dr, \qquad (112)$$

where the limit of integration $r_0(\phi)$ is determined by condition

$$\frac{r_0^2\cos^2\varphi}{a^2}+\frac{r_0^2\sin^2\varphi}{b^2} = 1. \qquad (113)$$

Fulfilling in (112) integration with respect to r and taking into account relation (113), we find

$$\iint_{\Sigma} \sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\,dx'dy' =$$
$$= -\frac{1}{3}\int_0^{2\pi}\left[1-r^2\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)\right]^{3/2}\Bigg|_{r=0}^{r=r_0}\frac{d\varphi}{\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}} =$$
$$= \frac{a^2b^2}{3}\int_0^{2\pi}\frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = \frac{2a^2b^2}{3}\int_0^{\pi}\frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi}, \tag{114}$$

since in the last definite integral the integrand has the period π. Substituting (60) into (114), we obtain

$$\iint_{\Sigma} \sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\,dx'dy' = \frac{2}{3}\pi ab. \tag{115}$$

Substituting (115) into (111), we find

$$a_1 = \frac{3P}{2\pi ab}. \tag{116}$$

Thus, we obtained four relations (108), (109), (110) and (116) for the determination of semiaxes of ellipse a and b, approach α and constant a_1, which enters into formula (88) for pressure p.

From relations (109) and (110) we find

$$\frac{E(e)-(1-e^2)K(e)}{(1-e^2)[K(e)-E(e)]} = \frac{A}{B},$$

or:

$$\left(\frac{1}{e^2}-1\right)\left[\frac{K(e)}{E(e)}-1\right] =$$
$$= \frac{B}{A+B}. \tag{117}$$

Equation (117) determines the eccentricity of the ellipse e according to the assigned ratio B/A. Let us note that ratio B/A and, consequently, the eccentricity of ellipse limiting the region of contact,

are determined only by the configuration of comrpessible bodies (see formulas (84) and (80)). Figure 51 shows the dependence of eccentricity e with respect to ratio B/A (let us remember that ratio B/A does not exceed unity).

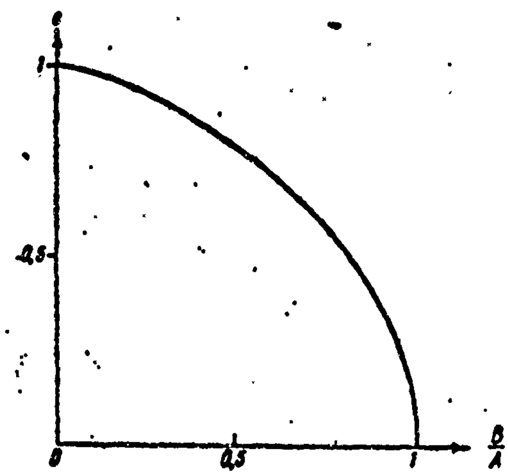

Fig. 51.

Substituting (116) into (110) and replacing in (110) the difference $1 - e^2$ by the ratio a^2/b^2, we find

$$\frac{3P}{2e^2b^3}[K(e) - E(e)] = \frac{B}{\vartheta_1 + \vartheta_2},$$

or according to (117)

$$\frac{3PE(e)}{2(1-e^2)b^3} = \frac{A+B}{\vartheta_1 + \vartheta_2}, \tag{118}$$

whence

$$b = \sqrt[3]{\frac{3E(e)}{2(1-e^2)} \frac{(\vartheta_1 + \vartheta_2)P}{A+B}}. \tag{119}$$

Detecting from equation (117) the eccentricity e, by formula (119) we find the semimajor axis of ellipse b. Knowing the semimajor axis of ellipse b and its eccentricity e, let us find th semiminor axis of ellipse a by formula

$$a = b\sqrt{1-e^2}. \tag{120}$$

Substituting (116) into (88), we obtain the final formula for pressure $p(x, y)$ in the region of contact

$$p(x,y)=\frac{3P}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}. \qquad (121)$$

Substituting (116) into (108), we obtain the formula for the approach α

$$\alpha=\frac{3}{2}K(e)\frac{(\vartheta_1+\vartheta_2)P}{b}. \qquad (122)$$

Formulas (117), (119), (120) (121) and (122) completely solve the examined contact problem of the theory of elasticity, successively determining the configuration of the region of contact, pressure in the region of contact and approach of compressible bodies with compression.

With $B = A$, i.e., in the case when the initial distance between points touching with compression, according to formulas (82) and (84) are equal to

$$f_1(x,y)+f_2(x,y)=Ar^2 \quad (r=\sqrt{x^2+y^2}), \qquad (123)$$

the eccentricity e of the ellipse, which limits the region of contact, is equal to zero according to graph 51. In this case $b = a$, and the region of contact turns into a circle of radius a. When $e = 0$, as one can see from formulas (96) and (101),

$$K(e)=E(e)=\frac{\pi}{2}. \qquad (124)$$

According to (124) formulas (119), (121) and (122) take the form

$$a=\sqrt[3]{\frac{3}{8}\pi\frac{(\vartheta_1+\vartheta_2)P}{A}}, \qquad (125)$$

$$p(x,y)=\frac{3P}{2\pi a^2}\sqrt{1-\frac{r^2}{a^2}}, \qquad (126)$$

$$\alpha=\frac{3}{4}\pi\frac{(\vartheta_1+\vartheta_2)P}{a}, \qquad (127)$$

and we arrive at the solution of the axisymmetrical contact problem of the theory of elasticity, obtained by us earlier in Chapter III.

The account we have given in this section of the solution of the contact problem belongs to Hertz[1]. In this solution an important fact is the assumption that in the expansion of (80) in the selection of the direction of coordinate axes x and y corresponding to conditions (81), not one of the coefficients $a_{11} + b_{11} = A$ and $a_{22} + b_{22} = B$ turns into zero. Actually, otherwise for the initial distance between points of the elastic bodies touching with compression, we would not have the right to take as the first approximation the expression (82). The special case when one of the coefficients A and B is equal to zero or both these coefficients are equal to zero was not examined by Hertz. For the case of radial symmetry we indicated complete solution of the problem on the compression of elastic bodies initially touching at a point regular for surfaces of both compressible bodies, not imposing any additional limitations on the configuration of these surfaces in the neighborhood of the point of initial contact. In the absence of radial symmetry such a complete solution of the contact problem of the theory of elasticity is associated with great mathematical difficulties. However, leaning on the property of the potential of the elliptic disk, indicated by us in § 1 of this chapter, it is possible to supplement the solution of Hertz by a number of other particular solutions of the contact problem of the theory of elasticity.

Let us consider the case when the expansion of (80) starts from the uniform polynomial of the fourth power, which takes with proper selection of the direction of coordinate axes x and y the form

$$Ax^4 + Bx^2y^2 + Cy^4 \quad (A > C).$$

Then, disregarding, due to the smallness of the region of contact, smalls of higher orders, in region Σ we will have

[1]Hertz H., Gesammelte Werke, Vol. 1, Leipzig, 1895, p. 155.

$$f_1(x, y)+f_2(x, y)=Ax^4+Bx^2y^2+Cy^4 \quad (A>C), \tag{128}$$

and condition (77) will take the form

$$\iint_\Sigma \frac{p(x', y')dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}=\frac{\alpha-Ax^4-Bx^2y^2-Cy^4}{\vartheta_1+\vartheta_2}. \tag{129}$$

If as region Σ we take again the part of plane xOy limited by the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the multiple integral standing in the left side of relation (129) will represent the potential of the elliptic disk $V(x, y, 0)$ with density $p(x', y')$. According to the formula (4) and (46), if for pressure $p(x', y')$ we take the expression

$$p(x', y')=a_1\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}+a_2\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)^{3/2}, \tag{130}$$

this multiple integral will equal

$$\begin{aligned}\iint_\Sigma \frac{p(x', y')dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}&=\pi ab\int_0^\pi \left\{\frac{1}{2}a_1\left[1-\frac{(x\sin\varphi-y\cos\varphi)^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}\right]+\right.\\ &\left.+\frac{3}{8}a_2\left[1-\frac{(x\sin\varphi-y\cos\varphi)^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}\right]^2\right\}\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}=\\ &=\pi ab\int_0^\pi\left[\frac{1}{2}a_1+\frac{3}{8}a_2-\left(\frac{1}{2}a_1+\frac{3}{4}a_2\right)\frac{(x\sin\varphi-y\cos\varphi)^2}{a^2\sin^2\varphi+b^2\cos^2\varphi}+\right.\\ &\left.+\frac{3}{8}a_2\frac{(x\sin\varphi-y\cos\varphi)^4}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^2}\right]\times\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}.\end{aligned} \tag{131}$$

In order that the polynomial in x and y in the right side of relation (131) is identically equal to the polynomial appearing in the right side of condition (129), it is necessary that

$$\frac{1}{2}a_1+\frac{3}{4}a_2=0,$$

whence

$$a_2 = -\frac{2}{3}a_1. \tag{132}$$

Substituting (132) into (130) and (131), we find that when

$$p(x', y') = a_1\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\left[1-\frac{2}{3}\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)\right], \tag{133}$$

$$\iint_{\Sigma}\frac{p(x', y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} = \frac{\pi}{4}aba_1\int_0^{\pi}\left[1-\frac{(x\sin\varphi-y\cos\varphi)^4}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^2}\right]\times$$

$$\times\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}} = \frac{\pi}{4}aba_1\int_0^{\pi}\left[1-\frac{x^4\sin^4\varphi-4x^3y\sin^3\varphi\cos\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^2}-\right.$$

$$\left.-\frac{6x^2y^2\sin^2\varphi\cos^2\varphi-4xy^3\sin\varphi\cos^3\varphi+y^4\cos^4\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^2}\right]+\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}}. \tag{134}$$

We find, further, by replacing ϕ by $\pi - \psi$

$$\int_0^{\pi}\frac{\sin^3\varphi\cos\varphi\,d\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^{5/2}} =$$

$$= \int_0^{\pi/2}\frac{\sin^3\varphi\cos\varphi\,d\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi^{5/2})}+\int_{\pi/2}^{\pi}\frac{\sin^3\varphi\cos\varphi\,d\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^{5/2}} =$$

$$= \int_0^{\pi/2}\frac{\sin^3\varphi\cos\varphi\,d\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^{5/2}}-\int_0^{\pi/2}\frac{\sin^3\psi\cos\psi\,d\psi}{(a^2\sin^2\psi+b^2\cos^2\psi)^{5/2}} = 0, \tag{135}$$

and analogously

$$\int_0^{\pi}\frac{\sin\varphi\cos^3\varphi\,d\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^{5/2}} = 0. \tag{136}$$

According to (135) and (136) relation (134) takes the form

$$\iint_{\Sigma}\frac{p(x', y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}} =$$

$$= \frac{\pi}{4}aba_1\int_0^{\pi}\left[1-\frac{x^4\sin^4\varphi+6x^2y^2\sin^2\varphi\cos^2\varphi+y^4\cos^4\varphi}{(a^2\sin^2\varphi+b^2\cos^2\varphi)^2}\right]\times$$

$$\times\frac{d\varphi}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}} =$$

$$= \frac{\pi a a_1}{4b^4}\int_0^{\pi}\left[b^4-\frac{x^4\sin^4\varphi+6x^2y^2\sin^2\varphi\cos^2\varphi+y^4\cos^4\varphi}{(1-e^2\sin^2\varphi)^2}\right]\times$$

$$\times\frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}},$$

where $e=\sqrt{1-\frac{a^2}{b^2}}$ — the eccentricity of the ellipse. Since in the obtained definite integral the integrand remains constant with replacement of ϕ by $\pi-\phi$, relation (137) can be given the form

$$\iint_S \frac{p(x',y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}=\frac{\pi a a_1}{2b^4}\int_0^{\pi/2}\left[b^4-\frac{x^4\sin^4\varphi+6x^2y^2\sin^2\varphi\cos^2\varphi+y^4\cos^4\varphi}{(1-e^2\sin^2\varphi)^2}\right]\times$$
$$\times\frac{d\varphi}{\sqrt{1-e^2\sin^2\varphi}}. \qquad (138)$$

Let us introduce the designation

$$\Delta(\varphi)=\sqrt{1-e^2\sin^2\varphi}. \qquad (139)$$

We find

$$\begin{aligned}
&x^4\sin^4\varphi+6x^2y^2\sin^2\varphi\cos^2\varphi+y^4\cos^4\varphi=\\
&=x^4\sin^4\varphi+6x^2y^2\sin^2\varphi(1-\sin^2\varphi)+y^4(1-2\sin^2\varphi+\sin^4\varphi)=\\
&=y^4+(6x^2y^2-2y^4)\sin^2\varphi+(x^4-6x^2y^2+y^4)\sin^4\varphi=\\
&=y^4+\frac{6x^2y^2-2y^4}{e^2}(1-\Delta^2)+\frac{x^4-6x^2y^2+y^4}{e^4}(1-2\Delta^2+\Delta^4)=\\
&=\frac{1}{e^4}[e^4y^4+6e^2x^2y^2-2e^2y^4+x^4-6x^2y^2+y^4-\\
&\quad(6e^2x^2y^2-2e^2y^4+2x^4-12x^2y^2+2y^4)\Delta^2+(x^4-6x^2y^2+y^4)\Delta^4]=\\
&=\frac{1}{e^4}\{x^4-6(1-e^2)x^2y^2+(1-e^2)^2y^4-2[x^4-3(2-e^2)x^2y^2+\\
&\qquad+(1-e^2)y^4]\Delta^2+(x^4-6x^2y^2+y^4)\Delta^4\}.
\end{aligned} \qquad (140)$$

Substituting (140) into (138), we obtain

$$\begin{aligned}
\iint_S \frac{p(x',y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}&=\frac{\pi q a_1}{2b^4e^4}\left\{(b^4e^4-x^4+6x^2y^2-y^4)\int_0^{\pi/2}\frac{d\varphi}{\Delta(\varphi)}+\right.\\
&+2[x^4-3(2-e^2)x^2y^2+(1-e^2)y^4]\int_0^{\pi/2}\frac{d\varphi}{\Delta^3(\varphi)}-\\
&\left.-[x^4-6(1-e^2)x^2y^2+(1-e^2)^2y^4]\int_0^{\pi/2}\frac{d\varphi}{\Delta^5(\varphi)}\right\}.
\end{aligned} \qquad (141)$$

According to the formulas (96) and (102)

$$\int_0^{\pi/2}\frac{d\varphi}{\Delta(\varphi)}=K(e),\quad \int_0^{\pi/2}\frac{d\varphi}{\Delta^3(\varphi)}=\frac{E(e)}{1-e^2}. \qquad (142)$$

In order to convert the third integral of integrals entering into formula (141), let us use identity

$$\frac{d}{d\varphi}\left(\frac{\sin\varphi\cos\varphi}{\Delta^3}\right)=\frac{1-2\sin^2\varphi}{\Delta^3}+\frac{3e^2\sin^2\varphi(1-\sin^2\varphi)}{\Delta^5}=\frac{e^2-2(1-\Delta^2)}{e^2\Delta^3}+$$
$$+\frac{3(1-\Delta^2)[e^2-(1-\Delta^2)]}{e^2\Delta^5}=\frac{2\Delta^2-(2-e^2)}{e^2\Delta^3}+\frac{-3\Delta^4+3(2-e^2)\Delta^2-3(1-e^2)}{e^2\Delta^5}=$$
$$=\frac{1}{e^2}\left(-\frac{1}{\Delta}+\frac{2(2-e^2)}{\Delta^3}-\frac{3(1-e^2)}{\Delta^5}\right). \tag{143}$$

Integrating both sides of identity (143) with respect to ϕ within limits of zero to $\pi/2$, we find

$$0=-\int_0^{\pi/2}\frac{d\varphi}{\Delta(\varphi)}+2(2-e^2)\int_0^{\pi/2}\frac{d\varphi}{\Delta^3(\varphi)}-3(1-e^2)\int_0^{\pi/2}\frac{d\varphi}{\Delta^5(\varphi)},$$

whence

$$\int_0^{\pi/2}\frac{d\varphi}{\Delta^5(\varphi)}=\frac{1}{3(1-e^2)}\left[2(2-e^2)\int_0^{\pi/2}\frac{d\varphi}{\Delta^3(\varphi)}-\int_0^{\pi/2}\frac{d\varphi}{\Delta(\varphi)}\right]=$$
$$=\frac{2(2-e^2)E(e)-(1-e^2)K(e)}{3(1-e^2)^2} \tag{144}$$

according to (142)

Substituting (142) and (144) into (141), we find

$$\iint_{\Sigma}\frac{p(x',y')\,dx'dy'}{\sqrt{(x-x')^2+(y-y')^2}}=\frac{\pi a a_1}{6b^4e^4(1-e^2)^2}\{3(b^4e^4-x^4+6x^2y^2-y^4)\times$$
$$\times(1-e^2)^2K(e)+6[x^4-3(2-e^2)x^2y^2+(1-e^2)y^4](1-e^2)E(e)-$$
$$-[x^4-6(1-e^2)x^2y^2+(1-e^2)^2y^4][2(2-e^2)E(e)-(1-e^2)K(e)]\}=$$
$$=\frac{\pi a a_1}{6b^4e^4(1-e^2)^2}\{3b^4e^4(1-e^2)^3K(e)+[-3(1-e^2)^2K(e)+$$
$$+6(1-e^2)E(e)-2(2-e^2)E(e)+(1-e^2)K(e)]x^4+$$
$$+6[3(1-e^2)^2K(e)-3(2-e^2)(1-e^2)E(e)+2(2-e^2)(1-e^2)E(e)-$$
$$-(1-e^2)^2K(e)]x^2y^2+[-3(1-e^2)^2K(e)+6(1-e^2)^2E(e)-$$
$$-2(2-e^2)(1-e^2)^2E(e)+(1-e^2)^3K(e)]y^4\}=\frac{\pi a a_1}{6b^4e^4(1-e^2)^2}\times$$
$$\times\{3b^4e^4(1-e^2)^2K(e)+[-(1-e^2)(2-3e^2)K(e)+2(1-2e^2)E(e)]x^4+$$
$$+6(1-e^2)[2(1-e^2)K(e)-(2-e^2)E(e)]x^2y^2+$$
$$+(1-e^2)^2[-(2+e^2)K(e)+2(1+e^2)E(e)]y^4\}. \tag{145}$$

Thus, if for pressure $p(x', y')$ we take expression (133), then multiple integral standing in the left side of condition (129) will be determined by formula (145). So that in this case condition (129) is fulfilled, it is sufficient that polynomials standing in right sides of relations (129) and (145) be identically equal to each other. By comparing coefficients of these two polynomials, we obtain conditions

$$\frac{\pi a a_1}{2}K(e)=\frac{\alpha}{\vartheta_1+\vartheta_2}, \tag{146}$$

$$\frac{\pi a a_1}{6b^4e^4(1-e^2)^2}\left[(1-e^2)(2-3e^2)K(e)-2(1-2e^2)E(e)\right]=\frac{A}{\vartheta_1+\vartheta_2}, \tag{147}$$

$$\frac{\pi a a_1}{b^4e^4(1-e^2)}\left[-2(1-e^2)K(e)+(2-e^2)E(e)\right]=\frac{B}{\vartheta_1+\vartheta_2}; \tag{148}$$

$$\frac{\pi a a_1}{6b^4e^4}\left[(2+e^2)K(e)-2(1+e^2)E(e)\right]=\frac{C}{\vartheta_1+\vartheta_2}. \tag{149}$$

Substituting (133) into (85), we obtain an additional condition

$$a_1\iint_{\Sigma}\sqrt{1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}}\left[1-\frac{2}{3}\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)\right]dx'dy'=P. \tag{150}$$

In turning from rectangular coordinates $x'y'$ to polar coordinates r, ϕ, i.e., assuming

$$x'=r\cos\varphi,\quad y'=r\sin\varphi,$$

we find

$$\iint_{\Sigma}\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)^{3/2}dx'dy'= \\ =\int_0^{2\pi}d\varphi\int_0^{r_0(\varphi)}\left[1-\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)r^2\right]^{3/2}r\,dr, \tag{151}$$

where the limit of integration $r_0(\phi)$ is determined by relation

$$\frac{r_0^2\cos^2\varphi}{a^2}+\frac{r_0^2\sin^2\varphi}{b^2}=1. \tag{152}$$

Fulfilling in (151) integration with respect to r and taking into account relation (152), we find

$$\iint_{\Omega}\left(1-\frac{x'^2}{a^2}-\frac{y'^2}{b^2}\right)^{3/2}dx'dy' =$$
$$= -\frac{1}{5}\int_0^{2\pi}\left[1-\left(\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}\right)r^2\right]^{5/2}\Bigg|_{r=0}^{r=r(\varphi)}\frac{d\varphi}{\frac{\cos^2\varphi}{a^2}+\frac{\sin^2\varphi}{b^2}} =$$
$$= \frac{a^2b^2}{5}\int_0^{2\pi}\frac{d\varphi}{a^2\sin^2\varphi+b^2\cos^2\varphi} = \frac{2}{5}\pi ab \qquad (153)$$

according to (60).

Substituting (115 and (153) into (150), we obtain

$$a_1\left(\frac{2}{3}\pi ab-\frac{2}{3}\cdot\frac{2}{5}\pi ab\right)=P,$$

whence

$$a_1=\frac{5P}{2\pi ab}. \qquad (154)$$

Thus, in order to obtain the solution of the examined contact problem, we must satisfy five conditions – (146), (147), (148), (149) and (154). At the same time at our disposal there are only four constants, the selection of which we can arrange in order – semiaxes of the ellipse a and b, approach α and coefficient a_1 in formula (133) for pressure p. Thus, aforementioned conditions impose a limitation on the assigned constants appearing in the formulation of the examined contact problem. Let us first discuss the character of this limitation. From conditions (147), (148) and (149) we find

$$\frac{[(2-e^2)K(e)-2(1-e^2)E(e)](1-e^2)^2}{(1-e^2)(2-3e^2)K(e)-2(1-2e^2)E(e)}=\frac{C}{A}, \qquad (155)$$

$$\frac{6[-2(1-e^2)K(e)+(2-e^2)E(e)](1-e^2)}{(1-e^2)(2-3e^2)K(e)-2(1-2e^2)E(e)}=\frac{B}{A}. \qquad (156)$$

Expressions standing in left sides of relations (155) and (156) depend only on eccentricity e. Thus, excluding from relations (155)

and (156) the eccentricity e, we obtain the connection between ratios B/A and C/A. In order to show the character of this connection, let us introduce the designation

$$\frac{C}{A}=k^2. \tag{157}$$

Since relations (155) and (156) express the ratio B/A as a function of ratio C/A, ratio B/A should be a certain definite function of parameter k introduced by us. Let us present this functional dependence in the form

$$\frac{B}{A}=2k[1-\delta(k)]. \tag{158}$$

In the table below a number of values of function $\delta(k)$ is given:

k	1	0,8303	0,6532	0,4663	0,2496
$\delta(k)$	0	0,0016	0,0065	0,0142	0,0351

As we see, function $\delta(k)$ over a wide range of the change in argument k obtains small values as compared to unity. Substituting values of constants B and C from (157) and (158) into (128), we find

$$f_1(x,y)+f_2(x,y)=A\{x^4+2k[1-\delta(k)]x^2y^2+k^2y^4\}. \tag{159}$$

Thus, so that we could arrive at a solution of the examined contact problem, the initial distance between points touching with compression $f_1(x,y)+f_2(x,y)$ should have expression (159), where A and k are constants subordinate to conditions

$$0<k\leq 1,\quad A>0, \tag{160}$$

and in other respects are arbitrary constants.

Since quantity $\delta(k)$ is small as compared to unity, formula (159) can be replaced by an approximate formula:

$$f_1(x, y)+f_2(x, y)=A(x^2+ky^2)^2. \qquad (161)$$

Figure 52 shows curves determined by equations

$$(x^2+ky^2)^2=c^4, \quad x^4+2k[1-\delta(k)]x^2y^2+k^2y^4=c^4$$

when k = 0.079 (in this case $\delta(k)$ = 0.124, and the eccentricity e, determined by equation (155), is equal to e = 0.975). The first of these equations determines the internal curve and the second the external curve on Fig. 52. As we see, even at values of k close to zero, and, accordingly, with eccentricity e close to unity, we obtain a fully satisfactory approximation, passing from the dependence (159) to the approximate dependence (161).

Fig. 52.

Thus, whereas Hertz showed the solution of the contact problem for the case

$$f_1(x, y)+f_2(x, y)=A(x^2+ky^2),$$

we actually arrived at the solution of the contact problem for the case

$$f_1(x, y)+f_2(x, y)=A(x^2+ky^2)^2.$$

Substituting (157) into (155), we obtain the equation

$$\frac{[(2+e^2)K(e)-2(1+e^2)E(e)](1-e^2)^2}{(1-e^2)(2-3e^2)K(e)-2(1-2e^2)E(e)} = k^3, \tag{162}$$

which determines the eccentricity e of the ellipse limiting the region of contact as a function of the assigned coefficient k. Figure 53 gives a graph of the dependence between the eccentricity e and coefficient k.

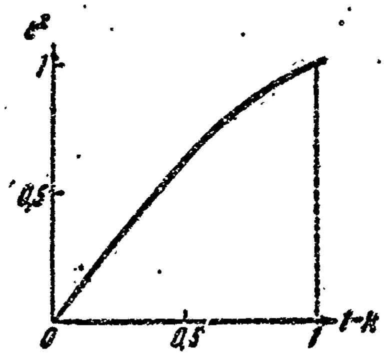

Fig. 53.

Substituting (154) into (147), let us obtain the relation

$$\frac{5P}{12b^3e^4(1-e^2)^2}[(1-e^2)(2-3e^2)K(e)-2(1-2e^2)E(e)] = \frac{A}{\theta_1+\theta_2},$$

from which it follows that

$$b = \sqrt[3]{\frac{5[(1-e^2)(2-3e^2)K(e)-2(1-2e^2)E(e)]}{12e^4(1-e^2)^2}\,\frac{(\theta_1+\theta_2)P}{A}}. \tag{163}$$

Thus, having determined from equation (162) the eccentricity e, by formula (163) we find the semimajor axis and by formula

$$a = b\sqrt{1-e^2} \tag{164}$$

the semiminor axis of the ellipse, which limits the region of contact. Substituting (154) into (133), let us obtain the final expression for pressure $p(x, y)$ in the region of contact:

$$p(x, y) = \frac{5P}{2\pi ab}\sqrt{1-\frac{x^2}{a^2}-\frac{y^2}{b^2}}\left[1-\frac{2}{3}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)\right]. \tag{165}$$

Substituting (154) into (146), we find the approach of elastic bodies with compression α:

$$\alpha = \frac{5K(e)}{4b}(\vartheta_1 + \vartheta_2)P. \tag{166}$$

Formulas (162), (163), (164), (165) and (166) completely solve the contact problem examined by us, successively determining the configuration of the region of contact, pressure in the region of contact and approach of compressible bodies with compression.

APPENDIX 1

REDUCTION OF CERTAIN ELLIPTIC INTEGRALS TO CANONICAL FORM

1. Let us first consider the definite integrals

$$c_n = \int_0^1 \frac{t^{n+\frac{1}{2}}\,dt}{\sqrt{1-t^2}}, \quad n = 0, 1, \ldots \tag{1}$$

Assuming in (1) $t = 1 - x^2$, we have

$$c_n = \sqrt{2}\int_0^1 \frac{(1-x^2)^{n+\frac{1}{2}}\,dx}{\sqrt{1-\frac{1}{2}x^2}},$$

or

$$c_n = \sqrt{2}\int_0^1 \frac{(1-x^2)^{n+1}\,dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}. \tag{2}$$

Introducing designations

$$X_n = \int_0^1 \frac{x^{2n}\,dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}, \quad n = 0, 1, \ldots, \tag{3}$$

we find

$$\left.\begin{aligned} c_0 &= \sqrt{2}(X_0 - X_1), \\ c_1 &= \sqrt{2}(X_0 - 2X_1 + X_2), \\ c_2 &= \sqrt{2}(X_0 - 3X_1 + 3X_2 - X_3), \text{ etc.} \end{aligned}\right\} \tag{4}$$

Thus, the calculation of definite integrals c_n [illegible] the calculation of definite integrals X_n. Using conven[illegible] designations

$$K(k)=\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}},\quad E(k)=\int_0^1 \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx \tag{5}$$

for complete elliptic integrals of the first and second kinds, we find

$$X_0=\int_0^1 \frac{dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}=K\left(\frac{\sqrt{2}}{2}\right), \tag{6}$$

$$\begin{aligned} X_1&=\int_0^1 \frac{x^2\,dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}=\\ &=2\left[\int_0^1 \frac{dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}-\int_0^1 \sqrt{\frac{1-\frac{1}{2}x^2}{1-x^2}}\,dx\right]=\\ &=2\left[K\left(\frac{\sqrt{2}}{2}\right)-E\left(\frac{\sqrt{2}}{2}\right)\right]. \end{aligned} \tag{7}$$

Let us use, further, the identity

$$\begin{aligned} &\frac{d}{dx}\left[x^{2n-3}\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}\right]=\\ &\qquad=\frac{2(2n-3)x^{2n-4}-3(2n-2)x^{2n-2}+(2n-1)x^{2n}}{2\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}. \end{aligned} \tag{8}$$

Integrating both sides of identity (8) with respect to x within limits of zero to unity, we find in accordance with designation (3)

$$0=\frac{1}{2}\left[2(2n-3)X_{n-2}-3(2n-2)X_{n-1}+(2n-1)X_n\right],$$

whence

$$X_n=3\frac{2n-2}{2n-1}X_{n-1}-2\frac{2n-3}{2n-1}X_{n-2},\qquad n=2,\ 3,\ \dots \tag{9}$$

Formula (9) makes it possible to express any integral X_n by integrals X_0 and X_1, which in turn are expressed according to formulas (6) and (7) by complete elliptic integrals of the first and second kinds.

In particular, from formula (9) it follows that

$$\left.\begin{aligned} X_2 &= \frac{2}{3}(3X_1 - X_0), \\ X_3 &= \frac{6}{5}(2X_2 - X_1) = \frac{2}{5}(9X_1 - 4X_0). \end{aligned}\right\} \qquad (10)$$

Substituting (10) into (4), we find

$$c_0 = \sqrt{2}(X_0 - X_1), \quad c_1 = \frac{\sqrt{2}}{3}X_0, \quad c_2 = \frac{2\sqrt{3}}{5}(X_0 - X_1). \qquad (11)$$

Substituting (6) and (7) into (11), we have

$$\left.\begin{aligned} c_0 &= \sqrt{2}\left[2E\left(\frac{\sqrt{2}}{2}\right) - K\left(\frac{\sqrt{2}}{2}\right)\right], \\ c_1 &= \frac{\sqrt{2}}{3}K\left(\frac{\sqrt{2}}{2}\right), \\ c_2 &= \frac{3\sqrt{2}}{5}\left[2E\left(\frac{\sqrt{2}}{2}\right) - K\left(\frac{\sqrt{2}}{2}\right)\right]. \end{aligned}\right\} \qquad (12)$$

From tables of complete elliptic integrals find (see, for example, Ya. Shpil'reyn, Tables of Special Functions, Part II):

$$K\left(\frac{\sqrt{2}}{2}\right) = 1{,}85407, \quad E\left(\frac{\sqrt{2}}{2}\right) = 1{,}35064. \qquad (13)$$

Substituting (13) into (12), we obtain

$$c_0 = 1{,}19814, \quad c_1 = 0{,}87403, \quad c_2 = 0{,}71888. \qquad (14)$$

We performed the calculation of integrals c_0 and c_2 in Chapter III (for integrals c_0 and c_2 we used designations J_1 and J_2)[1].

[1]See formula (150) of Chapter III.

2. Let us consider, further, the definite integral

$$J(\xi)=\int_0^1 \frac{t^{\frac{3}{2}}\,dt}{\sqrt{1-t^2}\,(t^2-\xi^2)}, \quad 0<\xi<1. \tag{15}$$

Assuming in (15) $t = 1 - x^2$, we obtain

$$J(\xi)=\sqrt{2}\int_0^1 \frac{(1-x^2)^2}{(1-x^2)^2-\xi^2}\,\frac{dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}. \tag{16}$$

We find further

$$\begin{aligned}\frac{(1-x^2)^2}{(1-x^2)^2-\xi^2}&=1+\frac{\xi^2}{(1-x^2)^2-\xi^2}=1+\frac{\xi}{2(1-x^2-\xi)}-\frac{\xi}{2(1-x^2+\xi)}=\\&=1+\frac{\xi}{2(1-\xi)}\left(1+\frac{x^2}{1-x^2-\xi}\right)-\frac{\xi}{2(1+\xi)}\left(1+\frac{x^2}{1-x^2+\xi}\right)=\\&=\frac{1}{1-\xi^2}+\frac{\xi}{2(1-\xi)}\,\frac{x^2}{1-x^2-\xi}-\frac{\xi}{2(1+\xi)}\,\frac{x^2}{1-x^2+\xi}.\end{aligned} \tag{17}$$

Substituting (17) into (16), we obtain

$$\begin{aligned}J(\xi)=\frac{\sqrt{2}}{2(1-\xi^2)}\Bigg[2\int_0^1 \frac{dx}{\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}&+\\+\xi\frac{1+\xi}{1-\xi}\int_0^1 \frac{x^2\,dx}{\left(1-\frac{x^2}{1-\xi}\right)\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}&-\\-\xi\frac{1-\xi}{1+\xi}\int_0^1 \frac{x^2\,dx}{\left(1-\frac{x^2}{1+\xi}\right)\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}&\Bigg].\end{aligned} \tag{18}$$

Introducing designation

$$\Pi(y,k)=\int_0^1 \frac{x^2\,dx}{(1-k^2x^2y^2)\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}, \tag{19}$$

and using designation (5), we can give to formula (18) the form

$$J(\xi)=\frac{\sqrt{3}}{3(1-\xi^2)}\left[2K\left(\frac{\sqrt{3}}{3}\right)+\xi\frac{1+\xi}{1-\xi}\Pi\left(\sqrt{\frac{3}{3-\xi}},\frac{\sqrt{3}}{3}\right)-\right.$$
$$\left.-\xi\frac{1-\xi}{1+\xi}\Pi\left(\sqrt{\frac{3}{1+\xi}},\frac{\sqrt{3}}{3}\right)\right]. \qquad (20)$$

The elliptic integral $\Pi(y, k)$ is expressed by elliptic integrals of the first and second kinds. This transformation is based on formulas of the addition of elliptic integrals, which we derive below.

3. In order to obtain formulas of the addition of elliptic integrals, let us examine the differential equation

$$\frac{dy}{dx}=-\frac{\Delta(y)}{\Delta(x)}, \qquad (21)$$

where

$$\Delta(x)=\sqrt{(1-x^2)(1-k^2x^2)} \qquad (22)$$

with the initial condition

$$y=z \text{ when } x=0 \quad (|z|<1). \qquad (23)$$

Dividing the variables in the differential equation (21), we obtain

$$\frac{dx}{\Delta(x)}=-\frac{dy}{\Delta(y)},$$

whence, in accordance with the initial condition (23),

$$\int_0^x\frac{dx}{\Delta(x)}=-\int_z^y\frac{dy}{\Delta(y)},$$

or

$$\int_0^x\frac{dx}{\Delta(x)}=\int_0^z\frac{dy}{\Delta(y)}-\int_0^y\frac{dy}{\Delta(y)}. \qquad (24)$$

Using for the elliptic integral of the first kind designation

$$\int_0^s \frac{dz}{\Delta(z)} = \int_0^s \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = F(s, k), \qquad (25)$$

we will be able to give to relation (24) the form

$$F(x, k) + F(y, k) = F(s, k). \qquad (26)$$

Thus, we obtained the solution of differential equation (21) in elliptic integrals. At the same time, as we will now show, for this differential equation the algebraic integral can be obtained.

If y as a function of x satisfies differential equation (21), then

$$\frac{d}{dx}(1-k^2x^2y^2) = -2k^2xy\left(y + x\frac{dy}{dx}\right) = -2k^2xy\left[y - x\frac{\Delta(y)}{\Delta(x)}\right],$$

$$\frac{d}{dx}[y\Delta(x) + x\Delta(y)] = y\Delta'(x) + \Delta(y) + [\Delta(x) + x\Delta'(y)]\frac{dy}{dx} =$$

$$= y\Delta'(x) + \Delta(y) - [\Delta(x) + x\Delta'(y)]\frac{\Delta(y)}{\Delta(x)} = y\Delta'(x) - x\Delta'(y)\frac{\Delta(y)}{\Delta(x)},$$

whence

$$\frac{d[y\Delta(x) + x\Delta(y)]}{d(1-k^2x^2y^2)} = \frac{y\Delta(x)\Delta'(x) - x\Delta(y)\Delta'(y)}{2k^2xy[x\Delta(y) - y\Delta(x)]}, \qquad (27)$$

according to (22)

$$\Delta^2(x) = (1-x^2)(1-k^2x^2) = 1 - (1+k^2)x^2 + k^2x^4,$$

whence

$$\Delta(x)\Delta'(x) = -(1+k^2)x + 2k^2x^3.$$

Thus,

$$y\Delta(x)\Delta'(x) - x\Delta(y)\Delta'(y) = y[2k^2x^3 - (1+k^2)x] - x[2k^2y^3 - (1+k^2)y] = 2k^2xy(x^2 - y^2). \qquad (28)$$

On the other hand

$$
\begin{aligned}
[x\Delta(y) - y\Delta(x)][x\Delta(y) + y\Delta(x)] &= x^2\Delta^2(y) - y^2\Delta^2(x) = \\
&= x^2[1 - (1 + k^2)y^2 + k^2y^4] - y^2[1 - (1 + k^2)x^2 + k^2x^4] = \\
&= x^2 + k^2x^2y^4 - y^2 - k^2x^4y^2 = (x^2 - y^2)(1 - k^2x^2y^2),
\end{aligned}
$$

whence

$$x\Delta(y) - y\Delta(x) = \frac{(x^2 - y^2)(1 - k^2x^2y^2)}{x\Delta(y) + y\Delta(x)}. \tag{29}$$

Substituting (28) and (29) into (27), we find

$$\frac{d[y\Delta(x) + x\Delta(y)]}{d(1 - k^2x^2y^2)} = \frac{y\Delta(x) + x\Delta(y)}{1 - k^2x^2y^2}. \tag{30}$$

Hence

$$
\left.
\begin{aligned}
&\frac{d[y\Delta(x) + x\Delta(y)]}{y\Delta(x) + x\Delta(y)} = \frac{d(1 - k^2x^2y^2)}{1 - k^2x^2y^2}, \\
&\ln[y\Delta(x) + x\Delta(y)] = \ln(1 - k^2x^2y^2) + \ln c, \\
&\frac{y\Delta(x) + x\Delta(y)}{1 - k^2x^2y^2} = c,
\end{aligned}
\right\} \tag{31}
$$

where c is an arbitrary constant.

Let us define now the arbitrary constant c in accordance with initial condition (23). Assuming in (31) $x = 0$, $y = z$, we find

$$c = z. \tag{32}$$

Substituting (32) into (31), we obtain

$$\frac{y\Delta(x) + x\Delta(y)}{1 - k^2x^2y^2} = z. \tag{33}$$

Thus, by integrating differential equation (21) according to the initial condition (23), we obtained the solution in two forms – in the form of relation (26) and in the form of relation (33). Thus, if numbers x, y and z satisfy relation (33), relation (26) takes place between elliptic integrals of the first kind with upper

limits x, y and z. We arrived at the formula of addition for the elliptic integral of the first kind. It is clear directly from relation (26) that this formula is accurate only when x and y satisfy the inequality

$$|F(x, k)+F(y, k)| \leqslant F(1, k)=K(k). \tag{34}$$

Let us now derive the formula of addition of the elliptic integral of the first kind when

$$K(k) \leqslant F(x, k)+ \quad (y, k) \leqslant 2K(k). \tag{35}$$

For this we replace the initial condition (23) by initial condition

$$y=1 \text{ when } x=z' \quad (0<z'\leqslant 1). \tag{36}$$

Integrating differential equation (21) according to the initial condition (36), we find

$$\int_{z'}^{x} \frac{dx}{\Delta(x)} = -\int_{1}^{y} \frac{dy}{\Delta(y)},$$

or:

$$\int_{0}^{x} \frac{dx}{\Delta(x)} - \int_{0}^{z'} \frac{dx}{\Delta(x)} = \int_{0}^{1} \frac{dy}{\Delta(y)} - \int_{0}^{y} \frac{dy}{\Delta(y)},$$

i.e.,

$$F(x, k)+F(y, k)=K(k)+F(z', k). \tag{37}$$

On the other hand, differential equation (21) has the solution (33), where z is the arbitrary constant. Let us define constant z in accordance with initial condition (36). Assuming in (33) $x = z'$ and $y = 1$, we find

$$\frac{\Delta(z')}{1-k^2z'^2}=z. \tag{38}$$

Thus, according to condition (35) instead of the formula of addition (26) we will have the formula of addition (37), where z' is connected with z by relation (38). Let us convert formula (37). From relation (38) we find

$$z^2 = \frac{1 - z'^2}{1 - k^2 z'^2}, \quad 1 - z^2 = \frac{(1 - k^2)\, z'^2}{1 - k^2 z'^2}, \quad 1 - k^2 z^2 = \frac{1 - k^2}{1 - k^2 z'^2},$$

$$\Delta(z) = \frac{(1 - k^2)\, z'}{1 - k^2 z'^2}, \quad -2k^2 z\, dz = \frac{2k^2 (1 - k^2) z'\, dz'}{(1 - k^2 z'^2)^2}, \quad dz = -\frac{(1 - k^2)\, z'\, dz'}{(1 - k^2 z'^2)\, \Delta(z')}$$

whence

$$\frac{dz}{\Delta(z)} = -\frac{dz'}{\Delta(z')}. \tag{39}$$

From (38) it follows that

$$z = 1 \text{ when } z' = 0. \tag{40}$$

Integrating (39) and taking into account (40), we find

$$\int_1^z \frac{dz}{\Delta(z)} = -\int_0^{z'} \frac{dz'}{\Delta(z')},$$

whence

$$F(z', k) = \int_0^{z'} \frac{dz'}{\Delta(z')} = \int_z^1 \frac{dz}{\Delta(z)} = \int_0^1 \frac{dz}{\Delta(z)} - \int_0^z \frac{dz}{\Delta(z)} = K(k) - F(z, k). \tag{41}$$

Substituting (41) into (37), we obtain the formula of addition

$$F(x, k) + F(y, k) = 2K(k) - F(z, k). \tag{42}$$

Let us derive now the formula of addition for elliptic integrals of the second kind.

Using identity (29), we can give to relation (33) the form

$$\frac{x^2 - y^2}{x\Delta(y) - y\Delta(x)} = z,$$

or

$$\frac{1-k^2x^2}{\Delta(x)} - \frac{1-k^2y^2}{\Delta(y)}\frac{\Delta(y)}{\Delta(x)} = k^2 z\left[y - x\frac{\Delta(y)}{\Delta(x)}\right]. \tag{43}$$

But if y as a function of x satisfies relation (33), then y satisfies the differential equation (21). Using relation (21), we can give to equality (43) the form

$$\frac{1-k^2x^2}{\Delta(x)}dx + \frac{1-k^2y^2}{\Delta(y)}dy = k^2 z(y\,dx + x\,dy). \tag{44}$$

Taking the initial conditions (23), we find from (44)

$$\int_0^x \frac{1-k^2x^2}{\Delta(x)}dx + \int_z^y \frac{1-k^2y^2}{\Delta(y)}dy = k^2 zxy,$$

or

$$\int_0^x \frac{1-k^2x^2}{\Delta(x)}dx + \int_0^y \frac{1-k^2y^2}{\Delta(y)}dy - \int_0^z \frac{1-k^2y^2}{\Delta(y)}dy = k^2 zxy. \tag{45}$$

Using for the elliptic integral of the second kind the designation

$$\int_0^x \frac{1-k^2x^2}{\Delta(x)}dx = \int_0^x \sqrt{\frac{1-k^2x^2}{1-x^2}}\,dx = E(x, k), \tag{46}$$

we will be able to give to relation (45) the form

$$E(x, k) + E(y, k) = E(z, k) + k^2 xyz. \tag{47}$$

Thus, if x, y and z satisfy relation (33) and inequality (34), for elliptic integrals of the second kind the formula of addition (47) takes place. If x and y satisfy inequality (35), then, using initial conditions (36), we find from (44)

$$\int_{z'}^{x} \frac{1-k^2x^2}{\Delta(x)}\,dx + \int_{1}^{y} \frac{1-k^2y^2}{\Delta(y)}\,dy = k^2 z(xy - z'),$$

or

$$\int_{0}^{x} \frac{1-k^2x^2}{\Delta(x)}\,dx - \int_{0}^{z'} \frac{1-k^2x^2}{\Delta(x)}\,dx + \int_{0}^{y} \frac{1-k^2y^2}{\Delta(y)}\,dy - \int_{0}^{1} \frac{1-k^2y^2}{\Delta(y)}\,dy = k^2 z(xy - z').$$

Using designations (46) and (5), we will be able to give to this relation the form

$$E(x, k) + E(y, k) = E(k) + E(z', k) + k^2 z(xy - z'). \tag{48}$$

From relations (38) and (39) it follows that

$$\frac{1-k^2z'^2}{\Delta(z')}\,dz' = -\frac{1-k^2}{1-k^2z^2}\,\frac{dz}{\Delta(z)}. \tag{49}$$

Integrating (49) and taking into account (40), we obtain

$$E(z', k) = -(1-k^2)\int_{1}^{z} \frac{dz}{(1-k^2z^2)\,\Delta(z)} = (1-k^2)\int_{z}^{1} \frac{dz}{(1-k^2z^2)\,\Delta(z)}. \tag{50}$$

We find further

$$\begin{aligned} k^2\frac{d}{dz}\left(\frac{z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}}\right) &= \frac{k^2\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} - \frac{k^2z^2}{\sqrt{(1-z^2)(1-k^2z^2)}} + \frac{k^4z^2\sqrt{1-z^2}}{(1-k^2z^2)^{3/2}} = \\ &= \frac{k^2(1-z^2)(1-k^2z^2) - k^2z^2(1-k^2z^2) + k^4z^2(1-z^2)}{(1-k^2z^2)\,\Delta(z)} = \\ &= \frac{k^2 - 2k^2z^2 + k^4z^4}{(1-k^2z^2)\,\Delta(z)} = \frac{1-k^2z^2}{\Delta(z)} - \frac{1-k^2}{(1-k^2z^2)\,\Delta(z)}. \end{aligned} \tag{51}$$

Using identity (51), we find according to (50)

$$E(z', k) = \int_z^1 \left[\frac{1-k^2z^2}{\Delta(z)} - k^2 \frac{d}{dz}\left(\frac{z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} \right) \right] dz =$$

$$= \int_0^1 \frac{1-k^2z^2}{\Delta(z)} dz - \int_0^z \frac{1-k^2z^2}{\Delta(z)} dz + \frac{k^2z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} =$$

$$= E(k) - E(z, k) + \frac{k^2z\sqrt{1-z^2}}{\sqrt{1-k^2z^2}}. \tag{52}$$

Further, according to (38)

$$\frac{\sqrt{1-z^2}}{\sqrt{1-k^2z^2}} = \sqrt{\frac{(1-k^2)z'^2}{1-k^2z'^2}} : \sqrt{\frac{1-k^2}{1-k^2z'^2}} = z'. \tag{53}$$

Substituting (53) into (52), we find

$$E(z', k) = E(k) - E(z, k) + k^2zz'. \tag{54}$$

Substituting (54) into (48), we obtain the formula of addition

$$E(x, k) + E(y, k) = 2E(k) - E(z, k) + k^2xyz. \tag{55}$$

Joining formulas of addition (26), (42), (47) and (55), we will finally have

$$\left.\begin{array}{l} F(x, k) + F(y, k) = F(z, k), \\ E(x, k) + E(y, k) = E(z, k) + k^2xyz \\ \qquad \text{when } |F(x, k) + F(y, k)| < K(k), \\ F(x, k) + F(y, k) = 2K(k) - F(z, k), \\ E(x, k) + E(y, k) = 2E(k) - E(z, k) + k^2xyz \\ \quad \text{when } K(k) \leqslant F(x, k) + F(y, k) < 2K(k), \end{array}\right\} \tag{56}$$

where

$$z = \frac{y\Delta(x) + x\Delta(y)}{1 - k^2x^2y^2}.$$

4. Let us turn to the transformation of the elliptic integral

$$\Pi(y, k) = \int_0^1 \frac{x^2 dx}{(1-k^2x^2y^2)\sqrt{(1-x^2)\left(1-\frac{1}{2}x^2\right)}}. \tag{57}$$

We will first assume that $0<y<1$. Since in this case when $0<x<1$

$$|F(x, k)+F(-y, k)|<K(k),$$

we will have, according to the formula of addition (56)

$$\left.\begin{aligned} &F(x, k)+F(-y, k)=F(\zeta, k), \\ &E(x, k)+E(-y, k)=E(\zeta, k)-k^2xy\zeta, \qquad 0<x<1, \end{aligned}\right\} \tag{58}$$

where

$$\zeta=\frac{-y\Delta(x)+x\Delta(y)}{1-k^2x^2y^2}. \tag{59}$$

But

$$F(-y, k)=\int_0^{-y}\frac{dy}{\Delta(y)}=-\int_0^{y}\frac{dy}{\Delta(y)}=-F(y, k), \tag{60}$$

and, analogously,

$$E(-y, k)=-E(y, k). \tag{61}$$

Thus, formulas (58) can be given the form

$$\left.\begin{aligned} &F(x, k)-F(y, k)=F(\zeta, k), \\ &E(x, k)-E(y, k)=E(\zeta, k)-k^2xy\zeta, \quad 0<x<1. \end{aligned}\right\} \tag{62}$$

Let us designate, further, by y' the number for which

$$F(y', k)+F(y, k)=K(k). \tag{63}$$

Then

$$\begin{aligned} &0<F(x, k)+F(y, k)<K \quad \text{when } 0<x<y', \\ &K<F(x, k)+F(y, k)<2K \quad \text{when } y'<x<1, \end{aligned}$$

and according to formulas of addition (56) we will have

$$\left.\begin{aligned} F(x, k) + F(y, k) &= F(z, k), \\ E(x, k) + E(y, k) &= E(z, k) + k^2 xyz, \quad 0 < x < y', \end{aligned}\right\} \tag{64}$$

$$\left.\begin{aligned} F(x, y) + F(y, k) &= 2K(k) - F(z, k), \\ E(x, k) + E(y, k) &= 2E(k) - E(z, k) + k^2 xyz, \quad y' < x < 1. \end{aligned}\right\} \tag{65}$$

From formulas (62), (64) and (65) we derive

$$\left.\begin{aligned} E(z, k) - E(\zeta, k) &= 2E(y, k) - k^2 xy(z+\zeta) \quad \text{when } 0 < x < y', \\ E(z, k) + E(\zeta, k) &= 2E(k) - 2E(y, k) + k^2 xy(z+\zeta) \\ &\qquad \text{when } y' < x < 1. \end{aligned}\right\} \tag{66}$$

Further, according to (56) and (59),

$$z + \zeta = \frac{2x'y}{1 - k^2x^2y^2}. \tag{67}$$

Substituting (67) into (66), we find

$$\left.\begin{aligned} k^2 y\Delta(y) \frac{x^2}{1-k^2x^2y^2} &= \frac{1}{2} E(\zeta, k) - \frac{1}{2} E(z, k) + E(y, k) \\ &\qquad \text{when } 0 < x < y', \\ k^2 y\Delta(y) \frac{x^2}{1-k^2x^2y^2} &= \frac{1}{2} E(\zeta, k) + \frac{1}{2} E(z, k) + E(y, k) - E(k) \\ &\qquad \text{when } y' < x < 1. \end{aligned}\right\} \tag{68}$$

Multiplying both sides of equalities (68) by $dx/\Delta(x)$ and integrating with respect to x, we find (considering that z and ζ depend on a):

$$\left.\begin{aligned} k^2 y\Delta(y) \int_0^{y'} \frac{x^2\,dx}{(1-k^2x^2y^2)\Delta(x)} &= \frac{1}{2}\int_0^{y'} \frac{E(\zeta, k)\,dx}{\Delta(x)} - \\ &\quad - \frac{1}{2}\int_0^{y'} \frac{E(z, k)\,dx}{\Delta(x)} + E(y, k)\int_0^{y'} \frac{dx}{\Delta(x)}, \\ k^2 y\Delta(y) \int_{y'}^{1} \frac{x^2\,dx}{(1-k^2x^2y^2)\Delta(x)} &= \frac{1}{2}\int_{y'}^{1} \frac{E(\zeta, k)\,dx}{\Delta(x)} + \\ &\quad + \frac{1}{2}\int_{y'}^{1} \frac{E(z, k)\,dx}{\Delta(x)} + [E(y, k) - E(k)]\int_{y'}^{1} \frac{dx}{\Delta(x)}, \end{aligned}\right\} \tag{69}$$

whence

$$k^2y\Delta(y)\int_0^1 \frac{x^2\,dx}{(1-k^2x^2y^2)^2\Delta(x)} = \frac{1}{2}\int_0^1 \frac{E(\zeta,k)\,dx}{\Delta(x)} - \frac{1}{2}\int_0^{y'} \frac{E(z,k)\,dx}{\Delta(x)} +$$

$$+\frac{1}{2}\int_{y'}^1 \frac{E(z,k)\,dx}{\Delta(x)} + E(y,k)\int_0^1 \frac{dx}{\Delta(x)} - E(k)\int_{y'}^1 \frac{dx}{\Delta(x)}. \qquad (70)$$

Differentiating with respect to x both sides of the first of relations (62), we find

$$\frac{1}{\Delta(x)} = \frac{1}{\Delta(\zeta)}\frac{d\zeta}{dx}, \quad 0 < x < 1,$$

whence

$$\frac{dx}{\Delta(x)} = \frac{d\zeta}{\Delta(\zeta)}, \qquad 0 < x < 1. \qquad (71)$$

Differentiating with respect to x both sides of the first of relations (64) and the first of relations (65), we find

$$\frac{1}{\Delta(x)} = \frac{1}{\Delta(z)}\frac{dz}{dx}, \quad 0 < x < y', \quad \frac{1}{\Delta(x)} = -\frac{1}{\Delta(z)}\frac{dz}{dx}, \quad y' < x < 1,$$

whence

$$\frac{dx}{\Delta(x)} = \frac{dz}{\Delta(z)}, \quad 0 < x < y', \quad \frac{dx}{\Delta(x)} = -\frac{dz}{\Delta(z)}, \quad y' < x < 1. \qquad (72)$$

further, from (59) and (56) we find

$$\left.\begin{array}{llll} \zeta = -y \text{ when } & x = 0, & \zeta = \dfrac{\Delta(y)}{1-k^2y^2} \text{ when } & x = 1, \\ z = y \text{ when } & x = 0, & z = \dfrac{y\Delta(y')+y'\Delta(y)}{1-k^2y^2y'^2} \text{ when } & x = y', \\ z = \dfrac{\Delta(y)}{1-k^2y^2} \text{ when } & x = 1. & & \end{array}\right\} \qquad (73)$$

Introducing designations $\frac{\Delta(y)}{1-k^2y^2} = y_1$, and $\frac{y\Delta(y')+y'\Delta(y)}{1-k^2y^2y'^2} = y_2$, we find in accordance with formulas (71), (72) and (73):

$$\int_0^1 \frac{E(\zeta, k)\,dz}{\Delta(z)} - \int_y^{y'} \frac{E(z, k)\,dz}{\Delta(z)} + \int_{y'}^1 \frac{E(z, k)\,dz}{\Delta(z)} =$$

$$= \int_{-y}^{y_1} \frac{E(\zeta, k)\,d\zeta}{\Delta(\zeta)} - \int_y^{y_1} \frac{E(z, k)\,dz}{\Delta(z)} - \int_{y_1}^{y_1} \frac{E(z, k)\,dz}{\Delta(z)} =$$

$$= \int_{-y}^{y_1} \frac{E(\zeta, k)\,d\zeta}{\Delta(\zeta)} - \int_y^{y_1} \frac{E(z, k)\,dz}{\Delta(z)} = \int_{-y}^{y} \frac{E(\zeta, k)\,d\zeta}{\Delta(\zeta)} = 0, \tag{74}$$

since function $\frac{E(\zeta,k)}{\Delta(\zeta)}$ according to (61) is odd. Further, according to (63)

$$\int_{y'}^1 \frac{dz}{\Delta(z)} = \int_0^1 \frac{dz}{\Delta(z)} - \int_0^{y'} \frac{dz}{\Delta(z)} = K(k) - F(y', k) = F(y, k). \tag{75}$$

Substituting (74) and (85) into (70), we find

$$k^2 y \Delta(y) \int_0^1 \frac{z^2\,dz}{(1 - k^2 z^2 y^2)\,\Delta(z)} = K(k)\,E(y, k) - E(k)\,F(y, k). \tag{76}$$

Thus, when $0 < y < 1$ we arrived at the relation

$$\Pi(y, k) = \int_0^1 \frac{z^2\,dz}{(1 - k^2 z^2 y^2)\Delta(z)} =$$

$$= \frac{K(k) \int_0^y \sqrt{\frac{1 - k^2 y^2}{1 - y^2}}\,dy - E(k) \int_0^y \frac{dy}{\sqrt{(1 - y^2)(1 - k^2 y^2)}}}{k^2 y \sqrt{(1 - y^2)(1 - k^2 y^2)}}. \tag{77}$$

Quite analogously, by using formulas of addition of elliptic integrals corresponding case $y > 1$, we can obtain relations

$$\Pi(y, k) = \frac{-K(k) \int_1^y \sqrt{\frac{1 - k^2 y^2}{y^2 - 1}}\,dy + E(k) \int_1^y \frac{dy}{\sqrt{(y^2 - 1)(1 - k^2 y^2)}}}{k^2 y \sqrt{(y^2 - 1)(1 - k^2 y^2)}}, \tag{78}$$

$$1 < y < \frac{1}{k},$$

$$\Pi(y, k) = \frac{-K(k) \int_{1/k}^y \sqrt{\frac{k^2 y^2 - 1}{y^2 - 1}}\,dy - E(k) \int_{1/k}^y \frac{dy}{\sqrt{(y^2 - 1)(k^2 y^2 - 1)}}}{k^2 y \sqrt{(y^2 - 1)(k^2 y^2 - 1)}}, \tag{79}$$

$$y > 1/k.$$

We restrict ourself to the formal derivation of formulas (78) and (79) from formula (77).

Assuming in the right side of formula (77)

$$\sqrt{1-y^2}=i\sqrt{y^2-1},$$

we obtain

$$\Pi(y,k)=\frac{-K(k)\int\limits_0^y\sqrt{\frac{1-k^2y^2}{y^2-1}}\,dy+E(k)\int\limits_0^y\frac{dy}{\sqrt{(y^2-1)(1-k^2y^2)}}}{k^2y\sqrt{(y^2-1)(1-k^2y^2)}}. \qquad (80)$$

When $1<y<\frac{1}{k}$ we have

$$\int\limits_0^y\sqrt{\frac{1-k^2y^2}{y^2-1}}\,dy=\int\limits_0^1\sqrt{\frac{1-k^2y^2}{y^2-1}}\,dy+\int\limits_1^y\sqrt{\frac{1-k^2y^2}{y^2-1}}\,dy=$$
$$=\int\limits_1^y\sqrt{\frac{1-k^2y^2}{y^2-1}}+\text{ the imaginary part}, \qquad (81)$$

and similarly

$$\int\limits_0^y\frac{dy}{\sqrt{(y^2-1)(1-k^2y^2)}}=\int\limits_1^y\frac{dy}{\sqrt{(y^2-1)(1-k^2y^2)}}+\text{ the imaginary part} \qquad (82)$$

Substituting (81) and (82) into (80, we find

$$\Pi(y,k)=\frac{-K(k)\int\limits_1^y\sqrt{\frac{1-k^2y^2}{y^2-1}}\,dz+E(k)\int\limits_1^y\frac{dy}{\sqrt{(y-1)(1-k^2y^2)}}}{k^2y\sqrt{(y^2-1)(1-k^2y^2)}}+$$
$$+\text{ the imaginary part}. \qquad (83)$$

But the left side in relation (83) is the real part. Consequently the imaginary part of the expression standing in the right side of relation (83) must be equal to zero, and we arrive at formula (78). Assuming in the right side of formula (78)

$$\sqrt{1-k^2y^2}=i\sqrt{k^2y^2-1},$$

we obtain

$$\Pi(y,k)=\frac{-K(k)\int\limits_1^y\sqrt{\frac{k^2y^2-1}{y^2-1}}\,dy-E(k)\int\limits_1^y\frac{dy}{\sqrt{(y^2-1)(k^2y^2-1)}}}{k^2y\sqrt{(y^2-1)(k^2y^2-1)}}. \tag{84}$$

When $y>\frac{1}{k}$ we have

$$\int\limits_1^y\sqrt{\frac{k^2y^2-1}{y^2-1}}\,dy=\int\limits_1^{1/k}\sqrt{\frac{k^2y^2-1}{y^2-1}}\,dy+\int\limits_{1/k}^y\sqrt{\frac{k^2y^2-1}{y^2-1}}\,dy=$$
$$=\int\limits_{1/k}^y\sqrt{\frac{k^2y^2-1}{y^2-1}}\,dy+\text{the imaginary part}, \tag{85}$$

and analogously

$$\int\limits_1^y\frac{dy}{\sqrt{(y^2-1)(k^2y^2-1)}}=\int\limits_{1/k}^y\frac{dy}{\sqrt{(y^2-1)(k^2y^2-1)}}+\text{the imaginary part}. \tag{86}$$

Substituting (85) and (86) into (84), we find

$$\Pi(y,k)=\frac{-K(k)\int\limits_{1/k}^y\sqrt{\frac{k^2y^2-1}{y^2-1}}\,dy-E(k)\int\limits_{1/k}^y\frac{dy}{\sqrt{(y^2-1)(k^2y^2-1)}}}{k^2y\sqrt{(y^2-1)(k^2y^2-1)}}+$$
$$+\text{the imaginary part}. \tag{87}$$

But the left side in relation (87) is the real part. Consequently, the imaginary part of the expression standing in the right side of relation (87) must be equal to zero, and we arrive at formula (79).

Thus, elliptic integral $\Pi(y, k)$ is expressed by the elliptic integrals of the first and second kinds.

5. Above we obtained for the elliptic integral

$$J(\xi)=\int\limits_0^1\frac{t^{3/2}\,dt}{\sqrt{1-t^2}\,(t^2-\xi^2)},\quad 0<\xi<1,$$

the expression (see formula (80))

$$J(\xi)=\frac{\sqrt{2}}{2(1-\xi^2)}\left[2K\left(\frac{\sqrt{2}}{2}\right)+\xi\frac{1+\xi}{1-\xi}\Pi\left(\sqrt{\frac{2}{1-\xi}},\ \frac{\sqrt{2}}{2}\right)-\right.$$
$$\left.-\xi\frac{1-\xi}{1+\xi}\Pi\left(\sqrt{\frac{2}{1+\xi}},\ \frac{\sqrt{2}}{2}\right)\right]. \tag{88}$$

Let us transform this expression, using the above formulas for the elliptic integral $\Pi(y,\ k)$. Since when $0<\xi<1$

$$\sqrt{\frac{2}{1-\xi}}>\sqrt{2},\quad 1<\sqrt{\frac{2}{1+\xi}}<\sqrt{2},$$

in the elliptic integral $\Pi\left(\sqrt{\frac{2}{1-\xi}},\ \frac{\sqrt{2}}{2}\right)$ $y>\frac{1}{k}$, and in integral $\Pi\left(\sqrt{\frac{2}{1+\xi}},\ \frac{\sqrt{2}}{2}\right)$ $1<y<\frac{1}{k}$. Thus, for the transformation of the first of these integrals we must use formula (79) and for second – formula (78). Assuming in formula (79) $y=\sqrt{\frac{2}{1-\xi}},\ k=\frac{\sqrt{2}}{2}$, and in formula (78) $y=\sqrt{\frac{2}{1+\xi}},\ k=\frac{\sqrt{2}}{2}$, we find

$$\Pi\left(\sqrt{\frac{2}{1-\xi}},\ \frac{\sqrt{2}}{2}\right)=\frac{-K\left(\frac{\sqrt{2}}{2}\right)\int\limits_{\sqrt{2}}^{\sqrt{\frac{2}{1-\xi}}}\sqrt{\frac{\frac{1}{2}y^2-1}{y^2-1}}\,dy}{\frac{1}{2}\sqrt{\frac{2}{1-\xi}\left(\frac{2}{1-\xi}-1\right)\left(\frac{1}{1-\xi}-1\right)}}-$$
$$-\frac{E\left(\frac{\sqrt{2}}{2}\right)\int\limits_{\sqrt{2}}^{\sqrt{\frac{2}{1-\xi}}}\frac{dy}{\sqrt{(y^2-1)\left(\frac{1}{2}y^2-1\right)}}}{\frac{1}{2}\sqrt{\frac{2}{1-\xi}\left(\frac{2}{1-\xi}-1\right)\left(\frac{1}{1-\xi}-1\right)}}, \tag{89}$$

$$\Pi\left(\sqrt{\frac{2}{1+\xi}},\ \frac{\sqrt{2}}{2}\right)=\frac{-K\left(\frac{\sqrt{2}}{2}\right)\int\limits_{1}^{\sqrt{\frac{2}{1+\xi}}}\sqrt{\frac{1-\frac{1}{2}y^2}{y^2-1}}\,dy}{\frac{1}{2}\sqrt{\frac{2}{1+\xi}\left(\frac{2}{1+\xi}-1\right)\left(1-\frac{1}{1+\xi}\right)}}+$$
$$+\frac{E\left(\frac{\sqrt{2}}{2}\right)\int\limits_{1}^{\sqrt{\frac{2}{1+\xi}}}\frac{dy}{\sqrt{(y^2-1)\left(1-\frac{1}{2}y^2\right)}}}{\frac{1}{2}\sqrt{\frac{2}{1+\xi}\left(\frac{2}{1+\xi}-1\right)\left(1-\frac{1}{1+\xi}\right)}}. \tag{90}$$

Assuming $y=\frac{\sqrt{2}}{\eta}$, we find

$$\int_{\sqrt{2}}^{\sqrt{\frac{2}{1-\xi}}} \sqrt{\frac{\frac{1}{2}y^2-1}{y^2-1}}\,dy = \int_{\sqrt{1-\xi}}^{1} \sqrt{\frac{1-\eta^2}{1-\frac{1}{2}\eta^2}}\,\frac{d\eta}{\eta^2}, \tag{91}$$

$$\int_{\sqrt{2}}^{\sqrt{\frac{2}{1-\xi}}} \frac{dy}{\sqrt{(y^2-1)\left(\frac{1}{2}y^2-1\right)}} = \int_{\sqrt{1-\xi}}^{1} \frac{d\eta}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}}. \tag{92}$$

Using identity

$$\frac{d}{d\eta}\left(\frac{1}{\eta}\sqrt{1-\eta^2}\sqrt{1-\frac{1}{2}\eta^2}\right) = -\frac{1}{\eta^2}\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)} -$$
$$-\sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}} - \frac{1}{2}\sqrt{\frac{1-\eta^2}{1-\frac{1}{2}\eta^2}} = -\sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}} -$$
$$-\frac{1}{\eta^2}\sqrt{\frac{1-\eta^2}{1-\frac{1}{2}\eta^2}}\left(1-\frac{1}{2}\eta^2+\frac{1}{2}\eta^2\right) = -\sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}} - \frac{1}{\eta^2}\sqrt{\frac{1-\eta^2}{1-\frac{1}{2}\eta^2}},$$

we find

$$\int_{\sqrt{1-\xi}}^{1} \sqrt{\frac{1-\eta^2}{1-\frac{1}{2}\eta^2}}\,\frac{d\eta}{\eta^2} =$$
$$= \int_{\sqrt{1-\xi}}^{1} \left\{-\sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}} - \frac{d}{d\eta}\left[\frac{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}}{\eta}\right]\right\} d\eta =$$
$$= -\int_{0}^{1} \sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}}\,d\eta + \int_{0}^{\sqrt{1-\xi}} \sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}}\,d\eta +$$
$$+\sqrt{\frac{[1-(1-\xi)]\left[1-\frac{1}{2}(1-\xi)\right]}{1-\xi}} =$$
$$= -E\left(\frac{\sqrt{2}}{2}\right) + E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) + \sqrt{\frac{\xi(1+\xi)}{2(1-\xi)}}. \tag{93}$$

Further

$$\int_{\sqrt{1-\xi}}^{1} \frac{d\eta}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} = \int_{0}^{1} \frac{d\eta}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} - \int_{0}^{\sqrt{1-\xi}} \frac{d\eta}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} = K\left(\frac{\sqrt{2}}{2}\right) - F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right). \tag{94}$$

Substituting (93) and (94) into (91) and (92), we find

$$\left.\begin{aligned} &\int_{\sqrt{2}}^{\sqrt{\frac{2}{1-\xi}}} \sqrt{\frac{\frac{1}{2}y^2-1}{y^2-1}}\,dy = -E\left(\frac{\sqrt{2}}{2}\right) + \\ &\qquad + E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) + \sqrt{\frac{\xi(1+\xi)}{2(1-\xi)}}, \\ &\int_{\sqrt{2}}^{\sqrt{\frac{2}{1-\xi}}} \frac{dy}{\sqrt{(y^2-1)\left(\frac{1}{2}y^2-1\right)}} = \\ &\qquad = K\left(\frac{\sqrt{2}}{2}\right) - F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right). \end{aligned}\right\} \tag{95}$$

Substituting (95) into (89), we obtain

$$\Pi\left(\sqrt{\frac{2}{1-\xi}}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}(1-\xi)^{3/2}}{\sqrt{\xi(1+\xi)}}\left[E\left(\frac{\sqrt{2}}{2}\right)F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - K\left(\frac{\sqrt{2}}{2}\right)E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right)\right] - (1-\xi)K\left(\frac{\sqrt{2}}{2}\right). \tag{96}$$

Assuming $y = \frac{1}{\sqrt{1-\frac{1}{2}\eta^2}}$, we find

$$\int_{1}^{\sqrt{\frac{2}{1+\xi}}} \sqrt{\frac{1-\frac{1}{2}y^2}{y^2-1}}\,dy = \frac{1}{2}\int_{0}^{\sqrt{1-\xi}} \frac{\sqrt{1-\eta^2}\,d\eta}{\left(1-\frac{1}{2}\eta^2\right)^{3/2}}. \tag{97}$$

$$\int_{1}^{\sqrt{\frac{2}{1+\xi}}} \frac{dy}{\sqrt{(y^2-1)\left(1-\frac{1}{2}y^2\right)}} =$$

$$= \int_{0}^{\sqrt{1-\xi}} \frac{d\eta}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} = F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right). \tag{98}$$

Using identity

$$\frac{d}{d\eta}\left(\frac{\eta\sqrt{1-\eta^2}}{\sqrt{1-\frac{1}{2}\eta^2}}\right) = \frac{\sqrt{1-\eta}}{\sqrt{1-\frac{1}{2}\eta^2}} + \frac{\frac{1}{2}\eta^2\sqrt{1-\eta^2}}{\left(1-\frac{1}{2}\eta^2\right)^{3/2}} -$$

$$- \frac{\eta^2}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} = \frac{\sqrt{1-\eta^2}}{\left(1-\frac{1}{2}\eta^2\right)^{3/2}} - \frac{\eta^2}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}},$$

we find

$$\int_{0}^{\sqrt{1-\xi}} \frac{\sqrt{1-\eta^2}\,d\eta}{\left(1-\frac{1}{2}\eta^2\right)^{3/2}} =$$

$$= \int_{0}^{\sqrt{1-\xi}} \frac{\eta^2\,d\eta}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} + \sqrt{\frac{(1-\xi)\,[1-(1-\xi)]}{1-\frac{1}{2}(1-\xi)}} =$$

$$= 2\int_{0}^{\sqrt{1-\xi}} \left[\frac{1}{\sqrt{(1-\eta^2)\left(1-\frac{1}{2}\eta^2\right)}} - \sqrt{\frac{1-\frac{1}{2}\eta^2}{1-\eta^2}}\right] d\eta + \sqrt{\frac{2\xi(1-\xi)}{1+\xi}} =$$

$$= 2F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - 2E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) + \sqrt{\frac{2\xi(1-\xi)}{1+\xi}}. \tag{99}$$

Substituting (99) into (97), we obtain

$$\int_{1}^{\sqrt{\frac{2}{1+\xi}}} \sqrt{\frac{1-\frac{1}{2}y^2}{y^2-1}}\,dy =$$

$$= F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) + \sqrt{\frac{\xi(1-\xi)}{2(1+\xi)}}. \tag{100}$$

Substituting (98) and (100) into (90), we find

$$\Pi\left(\sqrt{\frac{2}{1+\xi}}, \frac{\sqrt{2}}{2}\right) = \frac{\sqrt{2}(1+\xi)^{3/2}}{\sqrt{\xi(1-\xi)}}\left[K\left(\frac{\sqrt{2}}{2}\right)E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - \right.$$
$$- K\left(\frac{\sqrt{2}}{2}\right)F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) + E\left(\frac{\sqrt{2}}{2}\right)F\left(1-\xi, \frac{\sqrt{2}}{2}\right)\Big] -$$
$$-(1+\xi)K\left(\frac{\sqrt{2}}{2}\right). \tag{101}$$

Substituting (96) and (101) into (88), we obtain

$$J(\xi) = \frac{\sqrt{2}}{2(1-\xi^2)}\left\{2K\left(\frac{\sqrt{2}}{2}\right) + \right.$$
$$+\sqrt{2\xi(1-\xi^2)}\left[E\left(\frac{\sqrt{2}}{2}\right)F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - \right.$$
$$\left. - K\left(\frac{\sqrt{2}}{2}\right)E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - \xi(1+\xi)K\left(\frac{\sqrt{2}}{2}\right)\right] -$$
$$-\sqrt{2\xi(1-\xi^2)}\left[K\left(\frac{\sqrt{2}}{2}\right)E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - \right.$$
$$\left. - K\left(\frac{\sqrt{2}}{2}\right)F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) + E\left(\frac{\sqrt{2}}{2}\right)F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right)\right] +$$
$$\left. + \xi(1-\xi)K\left(\frac{\sqrt{2}}{2}\right)\right\} = \frac{\sqrt{2}}{2(1-\xi^2)}\left\{2(1-\xi^2)K\left(\frac{\sqrt{2}}{2}\right) + \right.$$
$$+\sqrt{2\xi(1-\xi^2)}\left[F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - \right.$$
$$\left. \left. -2E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right)\right]K\left(\frac{\sqrt{2}}{2}\right)\right\},$$

or finally

$$J(\xi) = K\left(\frac{\sqrt{2}}{2}\right)\left\{\sqrt{2} + \sqrt{\frac{\xi}{1-\xi^2}}\left[F\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right) - \right.\right.$$
$$\left.\left. -2E\left(\sqrt{1-\xi}, \frac{\sqrt{2}}{2}\right)\right]\right\}. \tag{102}$$

6. In Chapter II, solving the two-dimensional contact problem of the theory of elasticity for that case when the initial distance between points of compressible bodies touching with compression is proportional to $|x|^k$ (x— distance to the initial point of contact), arrived at the following formulas for pressure $p(x)$ and half-width of the region of contact a (formulas (103) and (104) of Chapter II):

$$p(x)=\frac{P}{\pi a}\sqrt{1-\frac{x^2}{a^2}}\,\frac{\int\limits_0^1\frac{t^k\,dt}{\sqrt{1-t^2}\left(t^2-\frac{x^2}{a^2}\right)}}{\int\limits_0^1\frac{t^k\,dt}{\sqrt{1-t^2}}},\quad 0<|x|<a, \tag{103}$$

$$a=\left[\frac{\pi P(\vartheta_1+\vartheta_2)}{2Ak\int\limits_0^1\frac{t^k dt}{\sqrt{1-t_2}}}\right]^{1/k}. \tag{104}$$

For the case $k={}^3/_2$, assuming $|x|=a\xi$ and using designations (1) and (15) for appropriate elliptic integrals, we have

$$p(a\xi)=\frac{P}{\pi a}\sqrt{1-\xi^2}\,\frac{J(\xi)}{C_1},\qquad 0<\xi<1, \tag{105}$$

$$a=\left[\frac{\pi P(\vartheta_1+\vartheta_2)}{3AC_1}\right]^{2/3}. \tag{106}$$

Substituting into (105) and (106) values of elliptic integrals C_1 and $J(\xi)$ from (12) and (102) we find

$$p(a\xi)=\frac{3P}{\pi a}\sqrt{1-\xi^2}\left\{1+\sqrt{\frac{\xi}{2(1-\xi^2)}}\left[F\left(\sqrt{1-\xi},\frac{\sqrt{2}}{2}\right)-\right.\right.$$
$$\left.\left.-2E\left(\sqrt{1-\xi},\frac{\sqrt{2}}{2}\right)\right]\right\}, \tag{107}$$

$$a=\left[\frac{\sqrt{2}\,\pi P(\vartheta_1+\vartheta_2)}{2AE\left(\frac{\sqrt{2}}{2}\right)}\right]^{2/3}. \tag{108}$$

Assuming $\xi=\cos^2\varphi$, and $t=\sin\psi$, we will have

$$F\left(\sqrt{1-\xi},\frac{\sqrt{2}}{2}\right)=\int\limits_0^{\sqrt{1-\xi}}\frac{dt}{\sqrt{(1-t^2)\left(1-\frac{1}{2}t^2\right)}}=\int\limits_0^{\varphi}\frac{d\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}},$$

$$E\left(\sqrt{1-\xi},\frac{\sqrt{2}}{2}\right)=\int\limits_0^{\sqrt{1-\xi}}\sqrt{\frac{1-\frac{1}{2}t^2}{1-t^2}}\,dt=\int\limits_0^{\varphi}\sqrt{1-\frac{1}{2}\sin^2\psi}$$

$$\left.\begin{aligned}&p=\frac{3P}{\pi a}\left[\sin\varphi\sqrt{1+\cos^2\varphi}+\right.\\&\left.+\frac{\sqrt{2}}{2}\cos^2\varphi\left(\int\limits_0^{\varphi}\frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}-2\int\limits_0^{\varphi}\sqrt{1-\frac{1}{2}\sin^2\varphi}\,d\varphi\right)\right],\\&\qquad\qquad\varphi=\arccos\sqrt{\frac{|x|}{a}}.\end{aligned}\right\} \tag{109}$$

We arrived, thus, at the formula convenient for calculations which determines the pressure $p(x)$ in the region of contact. In accordance with formula (109) a graph of the distribution of pressure in the region of contact, located for the examined case in Chapter II (Fig. 11) is plotted.

7. In Chapter III, by examining the axisymmetrical contact problem of the theory of elasticity for the case when the initial distance between points of compressible bodies, which touch upon compression, is proportional to $r^{5/2}$ (r — distance to the axis of axsymmetry), we arrived at the following formula for pressure $p(r)$ in the region of contact

$$p(r)=\frac{5P}{4\pi a^2} f\left(\frac{r}{a}\right), \quad 0<r<a, \tag{110}$$

where

$$f(\rho)=\sqrt{\rho}\int_{\rho}^{1}\frac{dt}{t\sqrt{t(1-t^2)}}, \quad 0<\rho<1, \tag{111}$$

see formulas (157) and (158) of Chapter III.

Assuming $t=\cos^2\psi$, $\rho=\cos^2\varphi$, we find

$$f(\cos^2\varphi)=\sqrt{2}\cos\varphi\int_{0}^{\varphi}\frac{d\psi}{\cos^2\psi\sqrt{1-\frac{1}{2}\sin^2\psi}}. \tag{112}$$

Using identity

$$\frac{d}{d\psi}\left(2\,\mathrm{tg}\,\psi\sqrt{1-\frac{1}{2}\sin^2\psi}\right)=\frac{2\sqrt{1-\frac{1}{2}\sin^2\psi}}{\cos^2\psi}-\frac{\sin^2\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}}=$$
$$=\frac{2-\sin^2\psi-\sin^2\psi\cos^2\psi}{\cos^2\psi\sqrt{1-\frac{1}{2}\sin^2\psi}}=\frac{1}{\cos^2\psi\sqrt{1-\frac{1}{2}\sin^2\psi}}+\frac{\cos^2\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}},$$

we find

$$\int_0^{\varphi}\frac{d\psi}{\cos^2\psi\sqrt{1-\frac{1}{2}\sin^2\psi}}=2\,\mathrm{tg}\,\varphi\sqrt{1-\frac{1}{2}\sin^2\varphi}-$$

$$-\int_0^{\varphi}\frac{\cos^2\psi\,d\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}}=\sqrt{2}\,\mathrm{tg}\,\varphi\sqrt{1+\cos^2\varphi}+$$

$$+\int_0^{\varphi}\frac{d\psi}{\sqrt{1-\frac{1}{2}\sin^2\psi}}-2\int_0^{\varphi}\sqrt{1-\frac{1}{2}\sin^2\psi}\,d\psi. \tag{113}$$

Substituting (113) into (112), we obtain

$$f(\cos^2\varphi)=2\sin\varphi\sqrt{1+\cos^2\varphi}+$$

$$+\sqrt{2}\cos\varphi\left(\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}-2\int_0^{\varphi}\sqrt{1-\frac{1}{2}\sin^2\varphi}\,d\varphi\right). \tag{114}$$

Assuming in (110) $r=a\cos^2\varphi$ and substituting into (110) the expression for $f(\cos^2\varphi)$ from (114), we find

$$\left.\begin{aligned}
p=\frac{5P}{2\pi a^2}\Bigg[\sin\varphi\sqrt{1+\cos^2\varphi}+\\
+\frac{\sqrt{2}}{2}\cos\varphi\left(\int_0^{\varphi}\frac{d\varphi}{\sqrt{1-\frac{1}{2}\sin^2\varphi}}-2\int_0^{\varphi}\sqrt{1-\frac{1}{2}\sin^2\varphi}\,d\varphi\right)\Bigg],\\
\varphi=\arccos\sqrt{\frac{r}{a}}.
\end{aligned}\right\} \tag{115}$$

Comparing formulas (115) with formulas (109), we are convinced of the fact that the distribution of pressure in the region of contact in the examined axisymmetrical contact problem is the same as that in the corresponding two-dimensional problem of the theory of elasticity.

APPENDIX 2

THE APPROXIMATE SOLUTION OF CERTAIN INTEGRAL EQUATIONS OF THE CONTACT PROBLEM

1. In Chapter II, by examining the problem on the compression of two circular cylinders, the radii of which are almost equal, we showed that the solution to this problem is reduced to the solution of integro-differential Prandtl equation from the theory of an airfoil of finite span and discussed the method of solution of this equation proposed by I. Vekua [Translator's note: name not verified]. Examining, further, the problem of pressure of a rigid stamp on the elastic half-plane taking into account surface changes of the elastic medium, we arrived at an analogous integro-differential equation, the approximate solution of which can also be obtained by the method of I. Vekua. After one of the functions entering into the integro-differential equation is replaced by the properly constructed approximate expression, following the method of I. Vekua, the solution of such an integro-differential equation in closed form can be obtained. Here we arrive, however, at the calculation of definite integrals, which are not expressed in elementary functions, and in connection with this for numerical calculations general methods of approximation of the solution of integral equations can be more convenient. Taking into account graphs illustrating the corresponding sections of Chapter II, we used the method of finite differences. This method consists in the fact that the unknown function is assumed variable not continuously, but by jumps. Dividing the interval of the change in the unknown function into n parts and assuming that in each of the obtained subintervals this function

maintains a constant value, we reduce the solution of the integral equation to the detecting of these n values of the unknown function. By proper selection of these values we can achieve that the integral equation is satisfied at n points of that interval in which this equation should be satisfied. We arrive, thus, to the solution of the system of n linear equations with n unknowns. Solving these equations, we will obtain the approximate expression for the unknown function in the form of a step function, changing by jumps. Constructing its graph and smoothing the jumps, we obtain finally a smooth curve, which depicts the approximate solution of the integral equation. Below we give the calculations made by us.

2. As we showed in Chapter II, § 7, in the case of the compression of two circular cylinders, the radii of which are almost equal, pressure $p(\phi)$ in the region of contact is determined by the integral equation [Chapter II, equation (31)]:

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\cos(\varphi-\varphi')\ln \operatorname{tg}\frac{|\varphi-\varphi'|}{2}\,d\varphi' - \\ -(\varkappa_1 r_1+\varkappa_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\sin|\varphi-\varphi'|\,d\varphi' + 2\vartheta_1 r_1\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi' = \\ =(r_2-r_1)(1-\cos\varphi)-\alpha\cos\varphi, \qquad -\varphi_0<\varphi<\varphi_0. \qquad (1)$$

In order to exclude from equation (1) the unknown constant α, in (1) we set $\phi = 0$. Let us obtain

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\cos\varphi'\ln \operatorname{tg}\frac{|\varphi'|}{2}\,d\varphi' - \\ -(\varkappa_1 r_1+\varkappa_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\sin|\varphi'|\,d\varphi' + 2\vartheta_1 r_1\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi' = -\alpha. \qquad (2)$$

Substituting α from (2) into (1), we have

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\left[\cos(\varphi-\varphi')\ln \operatorname{tg}\frac{|\varphi-\varphi'|}{2}-\cos\varphi\cos\varphi'\ln \operatorname{tg}\frac{|\varphi'|}{2}\right]d\varphi' - \\ -(\varkappa_1 r_1+\varkappa_2 r_2)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\left[\sin|\varphi-\varphi'|-\cos\varphi\sin|\varphi'|\right]d\varphi' + \\ +2\vartheta_1 r_1(1-\cos\varphi)\int_{-\varphi_0}^{\varphi_0} p(\varphi')\,d\varphi' = (r_2-r_1)(1-\cos\varphi), \quad -\varphi_0<\varphi<\varphi_0. \qquad (3)$$

Integral equation (3) jointly with condition

$$\int_{-\varphi_0}^{\varphi_0} p(\varphi)\cos\varphi\, d\varphi = \frac{P}{r_1}, \qquad (4)$$

where P – the compressing force, determines the angle ϕ_0 and pressure $p(\phi)$ in the region of contact $-\phi_0 < \phi < \phi_0$. Since function $p(\phi)$ should be due to even symmetry, we have

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{-\varphi_0}^{0} p(\varphi')\left[\cos(\varphi-\varphi')\ln \operatorname{tg}\frac{|\varphi-\varphi'|}{2} - \cos\varphi\cos\varphi'\ln \operatorname{tg}\frac{|\varphi'|}{2}\right]d\varphi' -$$
$$-(\varkappa_1 r_1+\varkappa_2 r_2)\int_{-\varphi_0}^{0} p(\varphi')\left[\sin|\varphi-\varphi'| - \cos\varphi\sin|\varphi'|\right]d\varphi' +$$
$$+2\vartheta_1 r_1(1-\cos\varphi)\int_{\varphi}^{0} p(\varphi')\,d\varphi' =$$
$$= 2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{0}^{\varphi_0} p(\varphi')\left[\cos(\varphi+\varphi')\ln \operatorname{tg}\frac{|\varphi+\varphi'|}{2} - \cos\varphi\cos\varphi'\ln \operatorname{tg}\frac{\varphi'}{2}\right]d\varphi' - (\varkappa_1 r_1+\varkappa_2 r_2)\int_{0}^{\varphi_0} p(\varphi')[\sin|\varphi+$$
$$+\varphi'| - \cos\varphi\sin\varphi']\,d\varphi' + 2\vartheta_1 r_1(1-\cos\varphi)\int_{0}^{\varphi} p(\varphi')\,d\varphi',$$

in virtue of which the integral equation (3) can be given the following form

$$2(\vartheta_1 r_1+\vartheta_2 r_2)\int_{0}^{\varphi_0} p(\varphi')\left[\cos(\varphi-\varphi')\ln \operatorname{tg}\frac{|\varphi-\varphi'|}{2} + \cos(\varphi+\varphi')\ln \operatorname{tg}\frac{|\varphi+\varphi'|}{2} - 2\cos\varphi\cos\varphi'\ln \operatorname{tg}\frac{\varphi'}{2}\right]d\varphi' -$$
$$-(\varkappa_1 r_1+\varkappa_2 r_2)\int_{0}^{\varphi_0} p(\varphi')[\sin|\varphi-\varphi'| + \sin|\varphi+\varphi'| -$$
$$-2\cos\varphi\sin\varphi']\,d\varphi' + 4\vartheta_1 r_1(1-\cos\varphi)\int_{0}^{\varphi_0} p(\varphi')\,d\varphi' =$$
$$= (r_2 - r_1)(1-\cos\varphi), \qquad -\varphi_0 < \varphi < \varphi_0. \qquad (5)$$

Let us divide the interval $(0, \phi_0)$ into n equal parts, and we will consider that in each of the obtained subintervals function $p(\phi)$ maintains a constant value:

$$p(\varphi) = p_k \text{ when } (k-1)\vartheta < \varphi < k\vartheta, \quad k = 1, 2, \ldots, n, \; \vartheta = \frac{\varphi_0}{n}. \tag{6}$$

Substituting $p(\phi)$ from (6) into (5), we obtain

$$\begin{aligned} &2(\theta_1 r_1 + \theta_2 r_2) \sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} \Big[\cos(\varphi - \varphi') \ln \operatorname{tg} \frac{|\varphi - \varphi'|}{2} + \\ &+ \cos(\varphi + \varphi') \ln \operatorname{tg} \frac{|\varphi + \varphi'|}{2} - 2 \cos \varphi \cos \varphi' \ln \operatorname{tg} \frac{\varphi'}{2} \Big] d\varphi' - \\ &-(\varkappa_1 r_1 + \varkappa_2 r_2) \sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} [\sin|\varphi - \varphi'| + \sin|\varphi + \varphi'| - \\ &-2 \cos\varphi \sin\varphi'] d\varphi' + 4\theta_1 r_1 (1 - \cos\varphi) \sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} d\varphi' = \\ &= (r_2 - r_1)(1 - \cos\varphi), \quad -\varphi_0 < \varphi < \varphi_0. \end{aligned} \tag{7}$$

In the right and left sides of relation (7) the odd functions of ϕ stand. Consequently, if relation (7) takes place for any positive value of ϕ, then it takes place for the corresponding (equal in absolute value) negative value of ϕ. Further, when $\phi = 0$ both sides of relation (7) turn into zero and, consequently, are equal to each other. Thus, if one were to determine $p_1, p_2, \ldots, p_n$ in such a manner that relation (7) is fulfilled when $\varphi = \vartheta, 2\vartheta, \ldots, n\vartheta$, then it will be fulfilled at all points $\varphi = l\vartheta\,(l = -n, \ldots, -1, 0, 1, \ldots, n)$.

Assuming in (7) $\varphi = l\vartheta\,(l = 1, 2, \ldots, n)$, we obtain the system of equations

$$\begin{aligned} &2(\theta_1 r_1 + \theta_2 r_2) \sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} \Big[\cos(l\vartheta - \varphi') \ln \operatorname{tg} \frac{|l\vartheta - \varphi'|}{2} + \\ &\cos(l\vartheta + \varphi') \ln \operatorname{tg} \frac{l\vartheta + \varphi'}{2} - 2 \cos l\vartheta \cos \varphi' \ln \operatorname{tg} \frac{\varphi'}{2} \Big] d\varphi' - \\ &-(\varkappa_1 r_1 + \varkappa_2 r_2) \sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} [\sin|l\vartheta - \varphi'| + \sin(l\vartheta + \varphi') - \\ &-2 \cos l\vartheta \sin\varphi'] d\varphi' + 4\theta_1 r_1 (1 - \cos l\vartheta) \sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} d\varphi' = (r_2 - r_1)(1 - \cos l\vartheta), \; l = 1, 2, \ldots, n. \end{aligned} \tag{8}$$

Let us turn to calculation of definite integrals appearing in equations (8). Let us find

when $k<l$

$$\int_{(k-1)\vartheta}^{k\vartheta} \cos(l\vartheta-\varphi')\ln \operatorname{tg}\frac{|l\vartheta-\varphi'|}{2}d\varphi' =$$

$$= \int_{(k-1)\vartheta}^{k\vartheta} \cos(l\vartheta-\varphi')\ln \operatorname{tg}\frac{l\vartheta-\varphi'}{2}d\varphi' =$$

$$= -\sin(l\vartheta-\varphi')\ln \operatorname{tg}\frac{l\vartheta-\varphi'}{2} - \varphi' \Big|_{\varphi'=(k-1)\vartheta}^{\varphi'=k\vartheta} =$$

$$= -\sin(l-k)\vartheta\ln \operatorname{tg}\frac{(l-k)\vartheta}{2} +$$

$$+\sin(l-k+1)\vartheta\ln \operatorname{tg}\frac{(l-k+1)\vartheta}{2} - \vartheta;$$

when $k>l+1$

$$\int_{(k-1)\vartheta}^{k\vartheta} \cos(l\vartheta-\varphi')\ln \operatorname{tg}\frac{|l\vartheta-\varphi'|}{2}d\varphi' =$$

$$= \int_{(k-1)\vartheta}^{k\vartheta} \cos(l\vartheta-\varphi')\ln \operatorname{tg}\frac{\varphi'-l\vartheta}{2}d\varphi' =$$

$$= -\sin(l\vartheta-\varphi')\ln \operatorname{tg}\frac{\varphi'-l\vartheta}{2} - \varphi' \Big|_{\varphi'=(k-1)\vartheta}^{\varphi'=k\vartheta} =$$

$$= -\sin(l-k)\vartheta\ln \operatorname{tg}\frac{(k-l)\vartheta}{2} + \sin(l-k+1)\vartheta\ln \operatorname{tg}\frac{(k-l-1)\vartheta}{2} - \vartheta,$$

$$\int_{(k-1)\vartheta}^{k\vartheta} \cos(l\vartheta+\varphi')\ln \operatorname{tg}\frac{l\vartheta+\varphi'}{2}d\varphi' =$$

$$= \sin(l\vartheta+\varphi')\ln \operatorname{tg}\frac{l\vartheta+\varphi'}{2} - \varphi' \Big|_{\varphi'=(k-1)\vartheta}^{\varphi'=k\vartheta} =$$

$$= \sin(l+k)\vartheta\ln \operatorname{tg}\frac{(l+k)\vartheta}{2} - \sin(l+k-1)\vartheta\ln \operatorname{tg}\frac{(l+k-1)\vartheta}{2} - \vartheta,$$

$$\int_{(k-1)\vartheta}^{k\vartheta} \cos\varphi'\ln \operatorname{tg}\frac{\varphi'}{2}d\varphi' = \sin\varphi'\ln \operatorname{tg}\frac{\varphi'}{2} - \varphi' \Big|_{\varphi'=(k-1)\vartheta}^{\varphi'=k\vartheta} =$$

$$= \sin k\vartheta\ln \operatorname{tg}\frac{k\vartheta}{2} - \sin(k-1)\vartheta\ln \operatorname{tg}\frac{(k-1)\vartheta}{2} - \vartheta.$$

Thus,

$$\int_{(k-1)\vartheta}^{k\vartheta} \Big[\cos(l\vartheta-\varphi')\ln \operatorname{tg}\frac{|l\vartheta-\varphi'|}{2} +$$

$$+\cos(l\vartheta+\varphi')\ln \operatorname{tg}\frac{l\vartheta+\varphi'}{2} - 2\cos l\vartheta\cos\varphi'\ln \operatorname{tg}\frac{\varphi'}{2}\Big]d\varphi' =$$

$$= \sin(l-k+1)\vartheta\ln \operatorname{tg}\frac{|l-k+1|\vartheta}{2} - \sin(l-k)\vartheta\ln \operatorname{tg}\frac{|l-k|\vartheta}{2} +$$

$$+\sin(l+k)\vartheta\ln \operatorname{tg}\frac{(l+k)\vartheta}{2} - \sin(l+k-1)\vartheta\ln \operatorname{tg}\frac{(l+k-1)\vartheta}{2} -$$

$$-2\vartheta - 2\cos l\vartheta\Big[\sin k\vartheta\ln \operatorname{tg}\frac{k\vartheta}{2} - \sin(k-1)\vartheta\ln \operatorname{tg}\frac{(k-1)\vartheta}{2} - \vartheta\Big]. \qquad (\cdot)$$

Further: when $k<l$

$$\int_{(k-1)\theta}^{k\theta} \sin|l\theta-\varphi'|\,d\varphi' = \int_{(k-1)\theta}^{k\theta} \sin(l\theta-\varphi')\,d\varphi' =$$
$$= \cos(l\theta-\varphi')\Big|_{\varphi'=(k-1)\theta}^{\varphi'=k\theta} = \cos(l-k)\theta-\cos(l-k+1)\theta;$$

when $k>l+1$

$$\int_{(k-1)\theta}^{k\theta} \sin|l\theta-\varphi'|\,d\varphi' = \int_{(k-1)\theta}^{k\theta} \sin(\varphi'-l\theta)\,d\varphi =$$
$$= -\cos(\varphi'-l\theta)\Big|_{\varphi'=(k-1)\theta}^{\varphi'=k\theta} = -\cos(l-k)\theta+\cos(l-k+1)\theta,$$
$$\int_{(k-1)\theta}^{k\theta} [\sin(l\theta+\varphi')-2\cos l\theta\sin\varphi']\,d\varphi' =$$
$$= -\cos(l\theta+\varphi')+2\cos l\theta\cos\varphi'\Big|_{\varphi'=(k-1)\theta}^{\varphi'=k\theta} =$$
$$= -\cos(l+k)\theta+\cos(l+k-1)\theta+2\cos l\theta\,[\cos k\theta-\cos(k-1)\theta].$$

Thus,

$$\int_{(k-1)\theta}^{k\theta} [\sin|l\theta-\varphi'|+\sin(l\theta+\varphi')-2\cos l\theta\sin\varphi']\,d\varphi' =$$
$$= \delta(l-k)\,[\cos(l-k)\theta-\cos(l-k+1)\theta]+$$
$$+\cos(l+k-1)\theta-\cos(l+k)\theta+2\cos l\theta\,[\cos k\theta-\cos(k-1)\theta], \qquad (10)$$

where

$$\left.\begin{aligned} &\delta(k)=1 \quad \text{when} \quad k>0,\\ &\delta(k)=-1 \quad \text{when} \quad k<-1. \end{aligned}\right\} \qquad (11)$$

Substituting (9) and (10) into (8), we obtain the system of equations

$$\sum_{k=1}^{n} p_k\Big\{2(\theta_1 r_1+\theta_2 r_2)\Big[\sin(l-k+1)\theta\ln\operatorname{tg}\frac{|l-k+1|\theta}{2}-$$
$$-\sin(l-k)\theta\ln\operatorname{tg}\frac{|l-k|\theta}{2}+\sin(l+k)\theta\ln\operatorname{tg}\frac{(l+k)\theta}{2}-$$
$$-\sin(l+k-1)\theta\ln\operatorname{tg}\frac{(l+k-1)\theta}{2}-2\theta\Big]+$$
$$+(\varkappa_1 r_1+\varkappa_2 r_2)\,\{\delta(l-k)\,[\cos(l-k+1)\theta-\cos(l-k)\theta]+$$
$$+\cos(l+k)\theta-\cos(l+k-1)\theta\}+$$

$$
\begin{aligned}
&+4\theta_1 r_1\theta-2\cos l\theta\left\{2(\theta_1 r_1+\theta_2 r_2)\left[\sin k\theta\ln\operatorname{tg}\frac{k\theta}{2}-\right.\right.\\
&\left.-\sin(k-1)\theta\ln\operatorname{tg}\frac{(k-1)\theta}{2}-\theta\right]+\\
&\left.+(\varkappa_1 r_1+\varkappa_2 r_2)[\cos k\theta-\cos(k-1)\theta]+2\theta_1 r_1\theta\right\}=\\
&\qquad=(r_2-r_1)(1-\cos l\theta),\qquad l=1,2,\ldots,n.
\end{aligned}\tag{12}
$$

Let us introduce designation

$$
\begin{aligned}
\Delta_k=2(\theta_1 r_1+\theta_2 r_2)&\left[\sin(k+1)\theta\ln\operatorname{tg}\frac{|k+1|\theta}{2}-\sin k\theta\ln\operatorname{tg}\frac{|k|\theta}{2}-\theta\right]+\\
&+(\varkappa_1 r_1+\varkappa_2 r_2)\,\delta(k)[\cos(k+1)\theta-\cos k\theta]+2\theta_1 r_1\theta.
\end{aligned}\tag{13}
$$

Then equations (12) can be given the form

$$
\sum_{k=1}^{n}p_k(\Delta_{l-k}+\Delta_{l+k-1}-2\cos l\theta\,\Delta_{k-1})=(r_2-r_1)(1-\cos l\theta),\tag{14}
$$
$$
l=1,2,\ldots,n,
$$

since

$$
\delta(l+k-1)=\delta(k-1)=1 \text{ when } k>1,\ l>1
$$

according to (11).

When $k>0$ we have

$$
\begin{aligned}
\Delta_k=2&(\theta_1 r_1+\theta_2 r_2)\times\\
&\times\left[\sin(k+1)\theta\ln\operatorname{tg}\frac{(k+1)\theta}{2}-\sin k\theta\ln\operatorname{tg}\frac{k\theta}{2}-\theta\right]+\\
&\qquad+(\varkappa_1 r_1+\varkappa_2 r_2)[\cos(k+1)\theta-\cos k\theta]+2\theta_1 r_1\theta.
\end{aligned}\tag{15}
$$

When $k>1$ we have also

$$
\begin{aligned}
\Delta_{-k}=2&(\theta_1 r_1+\theta_2 r_2)\times\\
&\times\left[-\sin(k-1)\theta\ln\operatorname{tg}\frac{(k-1)\theta}{2}+\sin k\theta\ln\operatorname{tg}\frac{k\theta}{2}-\theta\right]+\\
&+(\varkappa_1 r_1+\varkappa_2 r_2)[-\cos(k-1)\theta+\cos k\theta]+2\theta_1 r_1\theta,
\end{aligned}
$$

i.e.,

$$
\begin{aligned}
&+4\vartheta_2 r_1\vartheta - 2\cos l\vartheta\,\Big\{2(\vartheta_1 r_2+\vartheta_2 r_1)\Big[\sin k\vartheta \ln \operatorname{tg}\frac{k\vartheta}{2} - \\
&-\sin(k-1)\vartheta \ln \operatorname{tg}\frac{(k-1)\vartheta}{2} - \vartheta\Big] + \\
&+(\varkappa_1 r_2+\varkappa_2 r_1)[\cos k\vartheta - \cos(k-1)\vartheta] + 2\vartheta_2 r_1\vartheta\Big\} = \\
&\qquad = (r_2-r_1)(1-\cos l\vartheta), \qquad l = 1, 2, \ldots, n.
\end{aligned}
\tag{12}
$$

Let us introduce designation

$$
\begin{aligned}
\Delta_k = 2(\vartheta_1 r_2+\vartheta_2 r_1)\Big[\sin(k+1)\vartheta \ln \operatorname{tg}\frac{(k+1)\vartheta}{2} - \sin k\vartheta \ln \operatorname{tg}\frac{(k)\vartheta}{2} - \vartheta\Big] + \\
+(\varkappa_1 r_2+\varkappa_2 r_1)\,\delta(k)\,[\cos(k+1)\vartheta - \cos k\vartheta] + 2\vartheta_2 r_1\vartheta.
\end{aligned}
\tag{13}
$$

Then equations (12) can be given the form

$$
\sum_{k=1}^{n} p_k(\Delta_{l-k} + \Delta_{l+k-1} - 2\cos l\vartheta\,\Delta_{k-1}) = (r_2-r_1)(1-\cos l\vartheta), \tag{14}
$$
$$
l = 1, 2, \ldots, n,
$$

since

$$
\delta(l+k-1) = \delta(k-1) = 1 \text{ when } k>1,\ l>1
$$

according to (11).

When $k>0$ we have

$$
\begin{aligned}
\Delta_k = {}& 2(\vartheta_1 r_2+\vartheta_2 r_1)\times \\
&\times\Big[\sin(k+1)\vartheta \ln \operatorname{tg}\frac{(k+1)\vartheta}{2} - \sin k\vartheta \ln \operatorname{tg}\frac{k\vartheta}{2} - \vartheta\Big] + \\
&+(\varkappa_1 r_2+\varkappa_2 r_1)[\cos(k+1)\vartheta - \cos k\vartheta] + 2\vartheta_2 r_1\vartheta.
\end{aligned}
\tag{15}
$$

When $k>1$ we have also

$$
\begin{aligned}
\Delta_{-k} = {}& 2(\vartheta_1 r_2+\vartheta_2 r_1)\times \\
&\times\Big[-\sin(k-1)\vartheta \ln \operatorname{tg}\frac{(k-1)\vartheta}{2} + \sin k\vartheta \ln \operatorname{tg}\frac{k\vartheta}{2} - \vartheta\Big] + \\
&+(\varkappa_1 r_2+\varkappa_2 r_1)[-\cos(k-1)\vartheta + \cos k\vartheta] + 2\vartheta_2 r_1\vartheta,
\end{aligned}
$$

i.e.,

$$\Delta_{-k}=\Delta_{k-1} \text{ when } k\geqslant 1. \tag{16}$$

Introducing, further, designations

$$F(k)=2(\vartheta_1 r_1+\vartheta_2 r_2)\left(\sin k\theta \ln \operatorname{tg}\frac{k\theta}{2}-k\theta\right)+ + (\varkappa_1 r_1+\varkappa_2 r_2)\cos k\theta+2\vartheta_1 r_1 k\theta, \quad k\geqslant 0, \tag{17}$$

we can give to formula (15) the form

$$\Delta_k=F(k+1)-F(k), \quad k\geqslant 0. \tag{18}$$

Uniting formulas (14), (16), (17) and (18), we finally arrive at the following system of equations for the determination of unknowns p_1, p_2, ..., p_n:

$$\left.\begin{aligned}
&\sum_{k=1}^{n} p_k(\Delta_{l-k}+\Delta_{l+k-1}-2\cos l\theta\,\Delta_{k-1})=(r_2-r_1)(1-\cos l\theta),\\
&\qquad l=1, 2, \ldots, n,\\
&\text{where}\\
&\Delta_k=F(k+1)-F(k), \quad k\geqslant 0, \quad \Delta_{-k}=\Delta_{k-1}, \quad k\geqslant 1,\\
&F(k)=2(\vartheta_1 r_1+\vartheta_2 r_2)\left(\sin k\theta \ln \operatorname{tg}\frac{k\theta}{2}-k\theta\right)+\\
&\qquad +(\varkappa_1 r_1+\varkappa_2 r_2)\cos k\theta+2\vartheta_1 r_1 k\theta.
\end{aligned}\right\} \tag{19}$$

3. We will subsequently assume that compressible cylinders are made of one material, i.e.,

$$\vartheta_2=\vartheta_1, \quad \varkappa_2=\varkappa_1, \tag{20}$$

where $\vartheta_1=\frac{1-\mu^2}{\pi E}$, $\varkappa_1=\frac{(1+\mu)(1-2\mu)}{2E}$, E – elastic modulus and μ – Poisson's ratio of compressible bodies.

Since on the assumption that radii of compressible cylinders are almost equal, it is possible to also assume

$$r_1=r_2=r. \tag{21}$$

Then the expression for function $F(k)$ in formula (19) will take the form

$$F(k) = 4\vartheta_1 r \sin k\vartheta \ln \operatorname{tg} \frac{k\vartheta}{2} - 2\vartheta_1 r k\vartheta + 2\varkappa_1 r \cos k\vartheta,$$

or

$$F(k) = -2\vartheta_1 r f(k), \tag{22}$$

where

$$f(k) = -2 \sin k\vartheta \ln \operatorname{tg} \frac{k\vartheta}{2} + k\vartheta - c \cos k\vartheta, \tag{23}$$

$$c = \frac{\varkappa_1}{\vartheta_1} = \frac{\pi(1-2\mu)}{2(1-\mu)}. \tag{24}$$

Assuming further

$$\Delta_k = -2\vartheta r \delta_k, \tag{25}$$

we will have in accordance with (19) and (22)

$$\delta_k = f(k+1) - f(k), \quad k \geq 0, \ \delta_{-k} = \delta_{k-1}, \ k > 1. \tag{26}$$

Substituting (25) into equations for the determination of unknowns $p_1, \ldots, p_n$ (19), we obtain the equation

$$\sum_{k=1}^{n} p_k (2 \cos l\vartheta\, \delta_{k-1} - \delta_{l-k} - \delta_{l+k-1}) = \frac{\varepsilon}{2\vartheta r}(1 - \cos l\vartheta),$$
$$l = 1, 2, \ldots, n, \tag{27}$$

where

$$\varepsilon = r_2 - r_1. \tag{28}$$

Assum[illegible] in (27)

$$p_k = \frac{\varepsilon}{2\vartheta r} q_k, \quad k = 1, 2, \ldots, n, \tag{29}$$

we obtain the equation

$$\sum_{k=1}^{n} q_k (2\cos l\vartheta\, \delta_{k-1} - \delta_{l-k} - \delta_{l+k-1}) = 1 - \cos l\vartheta, \quad l = 1, 2, \ldots, n, \tag{30}$$

determining unknowns $q_1, q_2, \ldots, q_n$. Solving equation (30) for unknowns $q_1, q_2, \ldots, q_n$, then by formula (29) we will manage to find $p_1, p_2, \ldots, p_n$.

Below we give solutions of the system of equations (30) for three values of angle ϕ_0: 30°, 50° and 60°, taking Poisson's ratio μ equal to 0.3 and setting $n = 5$.

According to (24) when $\mu = 0.3$

$$c = \frac{2\pi}{7}.$$

According to (23) we will have

$$f(k) = -2\sin k\vartheta \ln \operatorname{tg}\frac{k\vartheta}{2} + k\vartheta - \frac{2\pi}{7}\cos k\vartheta. \tag{31}$$

Since $\vartheta = \frac{\varphi_0}{n}$, when $n = 5$ we will have $\vartheta = 6°$ when $\varphi_0 = 30°$, $\vartheta = 10°$ when $\varphi_0 = 50°$, $\vartheta = 12°$ when $\phi_0 = 60°$.

Given below are values of differences $\delta_k = f(k+1) - f(k)$, calculated in accordance with formula (31).

Table δ_k.

k \ φ_0	30°	50°	60°	k \ φ_0	30°	50°	60°
0	0,72610	1,03429	1,16484	5	0,16049	0,08526	0,03709
1	0,43974	0,45601	0,39035	6	0,12361	0,03467	−0,00601
2	0,33114	0,37052	0,36513	7	0,09251	0,00143	−0,02024
3	0,25921	0,24664	0,21612	8	0,06607	−0,01515	−0,00601
4	0,20464	0,15470	0,11000	9	0,04393	−0,01515	0,03709

Substituting from this table differences δ_k into equations (30) and using here relations $\delta_{-k}=\delta_{k-1}$ when $k>1$ (see (26)), we obtain equations

$$\varphi_0 = 30^\circ$$

$$0,27839q_1-0,18258q_2-0,04031q_3-0,02020q_4-0,01266q_5=0,00548,$$
$$0,64959q_1-0,12505q_2-0,28294q_3-0,09314q_4-0,05442q_5=0,02186,$$
$$0,79081q_1+0,19207q_2-0,25672q_3-0,35606q_4-0,14300q_5=0,04894,$$
$$0,86282q_1+0,31882q_2+0,04166q_3-0,34501q_4-0,41828q_5=0,08646,$$
$$0,89252q_1+0,37884q_2+0,14990q_3-0,05685q_4-0,41558q_5=0,13398.$$

$$\varphi_0 = 50^\circ$$

$$0,44682q_1-0,30970q_2-0,07288q_3-0,03944q_4-0,02720q_5=0,01520,$$
$$1,01730q_1-0,23598q_2-0,49264q_3-0,17774q_4-0,11446q_5=0,06030$$
$$1,17428q_1+0,25230q_2-0,47780q_3-0,64178q_4-0,28950q_5=0,13398$$
$$1,18326q_1+0,39606q_2-0,02304q_3-0,65788q_4-0,78212q_5=0,23397$$
$$1,08970q_1+0,45348q_2+0,10438q_3-0,22378q_4-0,82026q_5=0,35721$$

$$\varphi_0 = 60^\circ$$

$$0,52452q_1-0,37610q_2-0,09218q_3-0,05234q_4-0,03802q_5=0,0218$$
$$1,17460q_1-0,30334q_2-0,60872q_3-0,23260q_4-0,15814q_5=0,0864$$
$$1,30514q_1+0,51486q_2-0,61214q_3-0,81014q_4-0,39214q_5=0,1903$$
$$1,23408q_1+0,38780q_2-0,09570q_3-0,85638q_4-1,01262q_5=0,3308$$
$$1,01876q_1+0,38024q_2+0,02024q_3-0,36822q_4-1,09294q_5=0,5000$$

Solutions of these equations are reduced by us in the following table.

Table q_k.

φ_0 \ k	1	2	3	4	5
30°	0,1179	0,1132	0,1034	0,0870	0,0595
50°	[illegible]	[illegible]	[illegible]	0,2260	0,1753
60°	[illegible]	[illegible]	[illegible]	0,0212	0,4035

When $\mu = 0.3$ we have

$$\vartheta = \frac{1-\mu^2}{\pi E} = \frac{0,91}{\pi E}. \qquad (32)$$

Substituting (32) into (29), we find

$$p_k = -\frac{\pi}{1,82} q_k \frac{E\varepsilon}{r}. \qquad (33)$$

Substituting the found values of q_k into (33), we obtain values of p_k, reduced by us in the following table

Table p_k.

φ_0 \ k	1	2	3	4	5
30°	$0,2935 \frac{E\varepsilon}{r}$	$0,1954 \frac{E\varepsilon}{r}$	$0,1783 \frac{E\varepsilon}{r}$	$0,1502 \frac{E\varepsilon}{r}$	$0,1027 \frac{E\varepsilon}{r}$
50°	$0,6036 \frac{E\varepsilon}{r}$	$0,5790 \frac{E\varepsilon}{r}$	$0,5279 \frac{E\varepsilon}{r}$	$0,4419 \frac{E\varepsilon}{r}$	$0,3025 \frac{E\varepsilon}{r}$
60°	$1,4137 \frac{E\varepsilon}{r}$	$1,3844 \frac{E\varepsilon}{r}$	$1,2591 \frac{E\varepsilon}{r}$	$1,0723 \frac{E\varepsilon}{r}$	$0,6999 \frac{E\varepsilon}{r}$

Since on the assumption $p(\phi) = p_k$ when $(k-1)\vartheta < \varphi < k\vartheta$ (see (6)), the obtained table for values of 30°, 50°, and 60° of angle ϕ_0 enables plotting a graph of pressure p as a function of angle ϕ. Smoothing the obtained step graphs, we arrive at those distributions of pressure p in the region of contact which are represented on figures placed in Chapter II.

Assuming in relation (4)

$$p(\varphi) = p_k \text{ when } (k-1)\vartheta < \varphi < k\vartheta, \quad k = 1, 2, \ldots, n, \; r_1 = r,$$

we find

$$2\sum_{k=1}^{n} p_k \int_{(k-1)\vartheta}^{k\vartheta} \cos\varphi \, d\varphi = \frac{P}{2},$$

or

$$2\sum_{k=1}^{n} p_k[\sin k\theta - \sin(k-1)\theta] = \frac{P}{2}. \qquad (34)$$

Substituting (33) into (34), we obtain

$$\frac{P}{E\varepsilon} = \frac{\pi}{0{,}91}\sum_{k=1}^{n} q_k[\sin k\theta - \sin(k-1)\theta]. \qquad (35)$$

Using tables of values q_k, we find by formula (35)

$$\frac{P}{E\varepsilon} = 0{,}1676 \text{ when } \varphi_0 = 30^\circ,$$
$$\frac{P}{E\varepsilon} = 0{,}7722 \text{ when } \varphi_0 = 50^\circ,$$
$$\frac{P}{E\varepsilon} = 2{,}1118 \text{ when } \varphi_0 = 60^\circ.$$

Furthermore, obviously,

$$\frac{P}{E\varepsilon} = 0 \text{ when } \varphi_0 = 0,$$

since the compressing force should be equal to zero so that the contact is carried out at the point. Thus, we obtain four points for plotting the curve expressing the dependence of angle ϕ_0 on atio $P/E\varepsilon$. Plotted along these four points is the curve given in Chapter II which enables according to the difference in radii of cylinders ε, elastic modulus E and compressing force P finding angle ϕ_0 and thus determining dimensions of the region of contact.

Let us recall that if the region of contact is small, i.e., at small values of angle ϕ_0, for the solution of the examined contact problem it is possible to use the fundamental equation of the flat contact problem (as before we will assume that the compressible bodies are of the same material)

$$\frac{4(1-\mu^2)}{\pi E}\int_{-a}^{a} p(x')(\ln|x-x'| - \ln|x'|)\,dx' = Ax^2,$$

where

$$A = \frac{1}{2}\left(\frac{d^2 l}{dx^2}\right)_{x=0}, \tag{37}$$

$l(x)$ – initial distance between points touching with compression.

Jointly with condition

$$\int_{-a}^{a} p(x)\,dx = P \tag{38}$$

equation (36) determines the pressure $p(x)$ in the region of contact and half-width of this region a. As we indicated in Chapter II, the solution of this equation leads to formulas

$$p(x) = \frac{2P}{\pi a}\sqrt{1 - \frac{x^2}{a^2}}, \tag{39}$$

$$a = 2\sqrt{\frac{(1-\mu^2)P}{\pi E A}}. \tag{40}$$

In order to obtain representation about the accuracy which is ensured by the above used method of finite differences, we conducted by this method the solution of equation (36), having divided the half-interval of the change in function $p(x)$ - a into 5 equal parts and having assumed the pressure p to be constant in each of the obtained subintervals. Finally we arrived at the solution of the integral equation (36) depicted by the solid line on Fig. 54. The dashed line on the same figure shows the exact solution of this equation, plotted in accordance with formula (39). As we see, the curves differ from each other very little.

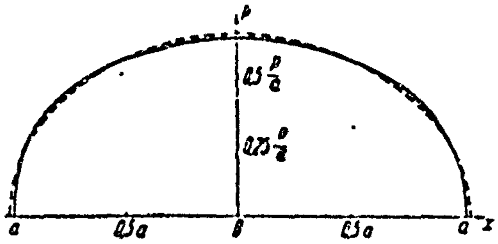

Fig. 54.

As we already noted above, at small values of angle ϕ_0 for the determination of the half-width of the region of contact a, i.e., quantity $r \sin \phi_0$, it is possible to use formula (40). Assuming in (40) $a = r \sin \phi_0$, we find

$$\varphi_0 = \arcsin 2 \sqrt{\frac{(1-\mu^2)P}{\pi E A r^2}}. \qquad (41)$$

The initial distance $l(x)$ between points touching with compression for the case of two circular cylinders will equal

$$l(x) = r_1 - \sqrt{r_1^2 - x^2} - (r_2 - \sqrt{r_2^2 - x^2}),$$

whence according to (37)

$$A = \frac{1}{2}\left(\frac{d^2 l}{dx^2}\right)_{x=0} = \frac{1}{2}\left(\frac{1}{r_1} - \frac{1}{r_2}\right) = \frac{r_2 - r_1}{2 r_1 r_2},$$

or, if one were to assume in the denominator $r_1 = r_2 = r$,

$$A = \frac{\varepsilon}{2r^2}. \qquad (42)$$

Substituting (42) into (41), we find

$$\varphi_0 = \arcsin 2 \sqrt{\frac{2(1-\mu^2)}{\pi} \frac{P}{E\varepsilon}}. \qquad (43)$$

In particular, when $\mu = 0.3$ formula (43) gives

$$\varphi_0 = \arcsin 2 \sqrt{\frac{1.82}{\pi} \frac{P}{E\varepsilon}}. \qquad (44)$$

In Chapter II we compared the dependence of angle ϕ_0 on the ratio $P/E\varepsilon$, obtained as a result of the solution of the exact integral equation of the problem (solid curve on the figure), with the dependence of ϕ_0 on $P/E\varepsilon$, determined by relation (44) (dashed curve on the same figure). As we see, for angle $\phi_0 = 30°$ formula (44), based on the assumption of the smallness of the region of contact, gives considerable error, and at larger values of angle ϕ_0 it is quite inapplicable.

In Chapter II we also compared the distribution pressure p in the region of contact, obtained as a result of the solution of the exact integral equation of the problem (solid curves on the figures), with the distribution of pressure determined by the well-known approximate formula

$$p(\varphi)=\frac{P\cos\varphi}{r(\sin\varphi_0\cos\varphi_0+\varphi_0)} \tag{45}$$

(dashed curves on the same figures).

Presenting formula (45) in the form

$$p(\varphi)=\frac{P\cos\varphi}{E\varepsilon(\sin\varphi_0\cos\varphi_0+\varphi_0)}\,\frac{E\varepsilon}{r},$$

and using values found above of ratio $P/E\varepsilon$ for angles $\phi_0 = 30°$, $\phi_0 = 50°$, and $\phi_0 = 60°$, we find

$$\left.\begin{aligned} p(\varphi)&=0{,}1732\cos\varphi\,\frac{E\varepsilon}{r}\ \text{when}\ \varphi_0=30°,\\ p(\varphi)&=0{,}5637\cos\varphi\,\frac{E\varepsilon}{r}\ \text{when}\ \varphi_0=50°,\\ p(\varphi)&=1{,}4267\cos\varphi\,\frac{E\varepsilon}{r}\ \text{when}\ \varphi_0=60°.\end{aligned}\right\} \tag{46}$$

In accordance with formulas (46) and dashed curves mentioned are plotted.

4. By examining the problem about the pressure of a rigid stamp on an elastic half-plane, taking into account surface changes of the elastic medium, we arrived in Chapter II at the solution of the integral equation [Chapter II, equation (286)]:

$$\pi p(a\xi)-c\int_{-1}^{1}p(a\tau)\ln|\tau-\xi|\,d\tau=\alpha,\quad -1<\xi<1, \tag{47}$$

where $p(x)$ – pressure under the stamp, a – half-width of the stamp, c – parameter depending on elastic constants and on surface properties of that elastic medium on which the stamp presses, and α – indefinite constant. Together with condition

$$\int_{-1}^{1} p(a\tau)\,d\tau = \frac{P}{a} \tag{48}$$

equation (47) uniquely determines the unknown function.

To plot graphs of the distribution of pressure under the stamp given in Chapter II, for different values of parameter c, we also used the method of finite differences. Below we give the calculations made by us.

Since function $p(x)$ is even, i.e., $p(-x) = p(x)$, we have

$$\int_{-1}^{0} p(a\tau)\ln|\tau-\xi|\,d\tau = \int_{0}^{1} p(-a\tau)\ln|-\tau-\xi|\,d\tau = \int_{0}^{1} p(a\tau)\ln|\tau+\xi|\,d\tau,$$

and equation (47) can be given the form

$$\pi p(a\xi) - c\int_{0}^{1} p(a\tau)[\ln|\tau-\xi| + \ln(\tau+\xi)]\,d\tau = \alpha, \quad 0<\xi<1. \tag{49}$$

Let us divide interval (0, a) into n equal parts, and in each of the obtained subintervals we will consider the pressure $p(x)$ to be constant:

$$p(x) = p_k \quad \text{when } (k-1)\frac{a}{n} < x < k\frac{a}{n}, \quad k=1, 2, \ldots, n. \tag{50}$$

Assuming in (50) $x = a\xi$, we find

$$p(a\xi) = p_k \quad \text{when } \frac{k-1}{n} < \xi < \frac{k}{n}, \quad k=1, 2, \ldots, n. \tag{51}$$

Let us define now quantities p_1, p_2, ..., p_n with such calculation that equation (49) is satisfied at n points $\xi = \frac{2l-1}{2n}$ (l = 1, 2, ..., n), i.e., so that there will be equalities

$$\pi p\left(a\frac{2l-1}{2n}\right) - c\int_{0}^{1} p(a\tau)\left[\ln\left|\tau - \frac{2l-1}{2n}\right| + \ln\left(\tau + \frac{2l-1}{2n}\right)\right]d\tau = \alpha,$$
$$l = 1, 2, \ldots, n. \tag{52}$$

Substituting (51) into (52), we obtain equations

$$\pi p_l - c\sum_{k=1}^{n} v_k \int\limits_{\frac{k-1}{n}}^{k/n}\left[\ln\left|\tau-\frac{2l-1}{2n}\right| + \ln\left(\tau+\frac{2l-1}{n}\right)\right]d\tau = x, \qquad (53)$$

$$l = 1, 2, \ldots, n.$$

When $k \geqslant l+1$ we find

$$\int\limits_{\frac{k-1}{n}}^{k/n} \ln\left|\tau-\frac{2l-1}{2n}\right| d\tau = \int\limits_{\frac{k-1}{n}}^{k/n} \ln\left(\tau-\frac{2l-1}{2n}\right) d\tau =$$
$$= \left(\tau-\frac{2l-1}{2n}\right)\ln\left(\tau-\frac{2l-1}{2n}\right) - \tau\Big|_{\frac{k-1}{n}}^{k/n} =$$
$$= \frac{2k-2l+1}{2n}\ln\frac{2k-2l+1}{2n} - \frac{2k-2l-1}{2n}\ln\frac{2k-2l-1}{2n} - \frac{1}{n};$$

when $k \leqslant l-1$ we find

$$\int\limits_{\frac{k-1}{n}}^{k/n} \ln\left|\tau-\frac{2l-1}{2n}\right| d\tau = \int\limits_{\frac{k-1}{n}}^{k/n} \ln\left(\frac{2l-1}{2n}-\tau\right) d\tau =$$
$$= \left(\tau-\frac{2l-1}{2n}\right)\ln\left(\frac{2l-1}{2n}-\tau\right) - \tau\Big|_{\frac{k-1}{n}}^{k/n} =$$
$$= \frac{2k-2l+1}{2n}\ln\frac{2l-2k-1}{2n} - \frac{2k-2l-1}{2n}\ln\frac{2l-2k+1}{2n} - \frac{1}{n};$$

and, finally, when $k = l$ we find

$$\int\limits_{\frac{k-1}{n}}^{k/n} \ln\left|\tau-\frac{2l-1}{2n}\right| d\tau = \int\limits_{\frac{l-1}{n}}^{\frac{2l-1}{2n}} \ln\left(\frac{2l-1}{2n}-\tau\right) d\tau +$$
$$+ \int\limits_{\frac{2l-1}{2n}}^{l/n} \ln\left(\tau-\frac{2l-1}{2n}\right) d\tau = \left(\tau-\frac{2l-1}{2n}\right)\ln\left(\frac{2l-1}{4n}-\tau\right) -$$
$$-\tau\Big|_{\frac{l-1}{n}}^{\frac{2l-1}{2n}} + \left(\tau-\frac{2l-1}{n}\right)\ln\left(\tau-\frac{2l-1}{2n}\right) - \tau\Big|_{\frac{2l-1}{2n}}^{l/n} = \frac{1}{n}\ln\frac{1}{2n} - \frac{1}{n}.$$

Thus,

$$\int_{\frac{k-1}{n}}^{k/n} \ln\left|\tau - \frac{2n-1}{2n}\right| d\tau = \frac{2k-2l+1}{2n} \ln \frac{|2k-2l+1|}{2n} - \\ - \frac{2k-2l-1}{2n} \ln \frac{|2k-2l-1|}{2n} - \frac{1}{n}, \quad k = 1, 2, \ldots, n. \tag{54}$$

Further

$$\int_{\frac{k-1}{n}}^{k/n} \ln\left(\tau + \frac{2l-1}{2n}\right) d\tau = \left(\tau + \frac{2l-1}{2n}\right) \ln\left(\tau + \frac{2l-}{2n}\right) - \tau \Big|_{\frac{k-1}{n}}^{k/n} = \\ = \frac{2k+2l-1}{2n} \ln \frac{2k+2l-1}{2n} - \frac{2k+2l-3}{2n} \ln \frac{2k+2l-3}{2n} - \frac{1}{n}, \\ k = 1, 2, \ldots, n. \tag{55}$$

Substituting (54) and (55) into (5.) and replacing for the convenience of calculations natural logarithms by common, we obtain the system of equations

$$\lambda p_l - \sum_{k=1}^{n} p_k \left(\frac{2k-2l+1}{2n} \lg \frac{|2k-2l+1|}{2n} - \right. \\ - \frac{2k-2l-1}{2n} \lg \frac{|2k-2l-1|}{2n} + \frac{2k+2l-1}{2n} \lg \frac{|2k+2l-1|}{2n} - \\ \left. - \frac{2k+2l-3}{2n} \lg \frac{2k+2l-3}{2n} - \frac{2}{n}\right) = \frac{\alpha M}{c}, \quad l = 1, 2, \ldots, n, \tag{56}$$

where

$$\lambda = \frac{\pi M}{c}, \quad M = \lg e = 0{,}43429. \tag{57}$$

Substituting (51) into (48), we find

$$\frac{1}{n} \sum_{k=1}^{n} p_k = \frac{P}{2n}. \tag{58}$$

Equations (56) and (58) determine unknowns p_1, p_2, ..., p_n and incidentally constant $\alpha M/c$. Determining p_1, p_2, ..., p_n, in accordance with formula (50) we obtain the approximate solution of integral equation (47) in the form of a step function.

5. Assuming in (56) $n = 5$ and taking into account (58), we obtain equations

$$(0{,}25686+\lambda)p_1+0{,}05052p_2-0{,}04843p_3-0{,}10934p_4-$$
$$-0{,}15396p_5=\frac{\alpha M}{c}-\frac{P}{a},$$
$$0{,}05052p_1+(0{,}15791+\lambda)p_2-0{,}01039p_3-0{,}09305p_4-$$
$$-0{,}14468p_5=\frac{\alpha M}{c}-\frac{P}{a},$$
$$-0{,}04843p_1-0{,}01039p_2+(0{,}11329+\lambda)p_3-0{,}04573p_4-$$
$$-0{,}12236p_5=\frac{\alpha M}{c}-\frac{P}{a},$$
$$-0{,}10934p_1-0{,}09305p_2-0{,}04573p_3+(0{,}08308+\lambda)p_4-$$
$$-0{,}07077p_5=\frac{\alpha M}{c}-\frac{P}{a},$$
$$-0{,}15396p_1-0{,}14468p_2-0{,}12236p_3-0{,}07077p_4+$$
$$+(0{,}06214+\lambda)p_5=\frac{\alpha M}{c}-\frac{P}{a}.$$

Excluding from these equations $\alpha M/c - P/a$, we obtain four equations

$$(0{,}20634+\lambda)p_1-(0{,}10739+\lambda)p_2-0{,}03804p_3-$$
$$-0{,}01629p_4-0{,}00928p_5=0,$$
$$0{,}09895p_1+(0{,}16830+\lambda)p_2-(0{,}12368+\lambda)p_3-$$
$$-0{,}04732p_4-0{,}02232p_5=0,$$
$$0{,}06091p_1+0{,}08266p_2+(0{,}15902+\lambda)p_3-$$
$$-(0{,}12971+\lambda)p_4-0{,}05159p_5=0,$$
$$0{,}04462p_1+0{,}05163p_2+0{,}07663p_3+(0{,}15475+\lambda)p_4-$$
$$-(0{,}13291+\lambda)p_5=0,$$

which together with equation (58)

$$p_1+p_2+p_3+p_4+p_5=\frac{5}{2}\frac{P}{a}$$

determine unknowns p_1, p_2, p_3, p_4 and p_5. In the table given below solutions of these equations for three values of parameter c are shown, namely: $c = 10$, $c = 1$ and $c = 0.1$ (i.e., according to (57) for $\lambda = 0.13644$, $\lambda = 1.3644$ and $\lambda = 13.644$).

Table p_k.

c \ k	1	2	3	4	5
10	$0{,}3658 \frac{P}{a}$	$0{,}3827 \frac{P}{a}$	$0{,}4239 \frac{P}{a}$	$0{,}5137 \frac{P}{a}$	$0{,}8121 \frac{P}{a}$
1	$0{,}4558 \frac{P}{a}$	$0{,}4645 \frac{P}{a}$	$0{,}4836 \frac{P}{a}$	$0{,}5171 \frac{P}{a}$	$0{,}5791 \frac{P}{a}$
0,1	$0{,}4913 \frac{P}{a}$	$0{,}4966 \frac{P}{a}$	$0{,}4982 \frac{P}{a}$	$0{,}5025 \frac{P}{a}$	$0{,}5096 \frac{P}{a}$

Having plotted in accordance with formula (50) and the above table p_k graphs of functions $p(x)$ and smoothed them, we obtain for $c = 10$, $c = 1$ and $c = 0.1$ distribution curves of pressure under the stamp, given in Chapter II.

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13. ABSTRACT

In the book Chapter I is devoted to methods of the solution of fundamental equations of the contact problem. An account of certain works of Soviet mathematicians is given concerning the two-dimensional contact problem of the theory of elasticity, including the author's works, part of which has been published for the first time. These include a new formulation of the problem on the pressure of a stamp on an elastic half-plane and the periodic contact problem discussed in Chapter II. The author makes an attempt to calculate surface deformations in Chapter II, which up till now have not been calculated in the theory of the contact problem. Chapter III gives a number of new solutions of an axisymmetric contact problem of the theory of elasticity. Together with the classical solutions, a number of new solutions belonging to the authors is given in Chapter IV.

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